Viscosity Solutions of Systems of Variational Inequalities with Interconnected Bilateral Obstacles.

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Abstract

We study a general class of nonlinear second-order variational inequalities with interconnected bilateral obstacles, related to a multiple modes switching game. Under rather weak assumptions, using systems of penalized unilateral backward SDEs, we construct a continuous viscosity solution of polynomial growth. Moreover, we establish a comparison result which in turn yields uniqueness of the solution.

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1 Introduction

In this paper we study systems of variational inequalities with interconnected lower and upper obstacles. This type of inequalities arises as the Bellman-Isaacs equation in a multiple modes switching game between two players. Besides their classical fields of applications, multiple modes switching games are attracting a lot of interest in the management of power plants (see Bernhart (2011) ([3]) and Perninge (2011) ([27]), where they are successfully used to design optimal stopping and starting strategies for power flow control through activation of regulating bids on a regulated power market.

The objective of this work is to establish existence and uniqueness of a continuous viscosity solution of the following system of variational inequalities with oblique reflection:

$$\begin{cases} \min\big\{\big(v^{ij}-L^{ij}[\vec{v}]\big)(t,x),\ \max\big\{\big(v^{ij}-U^{ij}[\vec{v}]\big)(t,x), \\ \qquad -\partial_t v^{i,j}(t,x)-\mathcal{L}v^{ij}(t,x)-f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x)D_xv^{ij}(t,x))\big\}\big\}=0, \\ v^{ij}(T,x)=h^{ij}(x), \end{cases} \tag{1.1}$$

for every pair (i,j) in the finite set of modes $\Gamma^1\times\Gamma^2$, where, for any $(t,x)\in[0,T]\times\mathbb{R}^k$,

$$\mathcal{L}\varphi(t,x):=b(t,x)D_x\varphi(t,x)+\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(t,x)D^2_{xx}\varphi(t,x)],$$

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and to any solution $\vec{v} = (v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ we associate the lower obstacle operator

$$L^{ij}[\vec{v}](t,x) := \max_{k\in\Gamma^1,\, k\neq i} \left(v^{kj} - \underline{g}_{ik}\right)(t,x),$$

and the upper obstacle operator

$$U^{ij}[\vec{v}](t,x) := \min_{l\in\Gamma^2,\, l\neq j} \left(v^{il} + \bar{g}_{jl}\right)(t,x),$$

where, $\underline{g}_{ik}$ (resp. $\bar{g}_{jl}$) stands for the switching cost incurred when the first (resp. second) player decides to switch from mode i to mode k (resp. from mode j to mode l). Finally, the function f^{ij} stands for the instantaneous payoff when the first player is in mode i and the second one in mode j.

The system (1.1) and related switching games have been studied by several authors. The most recent work discussing this topic includes the papers by Hu and Tang (2008) ([21]) and Tang and Hou (2007) ([28]) (see also the references therein) which deal with switching games related to (1.1), when the switching costs do not depend on the state variable. To the best of our knowledge, Ishii and Koike (1991) ([18]) are the latest most general existence and uniqueness results for the system (1.1), for state-dependent switching costs. They derive existence and uniqueness of viscosity solutions of the elliptic version of (1.1) in a bounded domain of $\mathbb{R}^k$ whose boundary is of class $\mathcal{C}^2$, when the so-called *Fichera functions* are strictly negative (see [18], Proposition 4.3).

The main result of the present paper, which is given in Theorem 5.4, establishes existence and uniqueness of a continuous viscosity solution of the system (1.1), when the state space is the whole $\mathbb{R}^k$ and under rather weak assumptions on the involved coefficients. Our approach is probabilistic and makes use of penalization schemes that allow us to connect the related penalized PDEs with systems of reflected backward SDEs with unilateral interconnected obstacles which, for instance, have been studied in [7], [11], [17] or [20]. With the help of these sequences of solutions of reflected BSDEs and their connection with PDEs, via Feynman-Kac's formula, we are able to construct in Propositions 5.1 and 5.2 both a viscosity subsolution and a supersolution for the system (1.1). Relying next both on the comparison result established in Theorem 4.2 and adapting the Perron's method we construct a solution for (1.1) which is therefore unique. Finally, using again the uniqueness result, we identify the limit of the penalized decreasing scheme as the solution of the system (1.1).

We made this detour instead of trying to solve a related system of reflected BSDEs with interconnected bilateral obstacles, as one would expect, simply because they satisfy neither the so-called Mokobodski condition nor the condition of strict separation of the two obstacles, which would guarantee existence and uniqueness of the solutions of the system of BSDEs, since these obstacles depend on the solution. The structure of these bilateral obstacles suggests a rather new type of conditions to guarantee existence and uniqueness result for the related system of reflected BSDEs. This problem is beyond the scope of the present paper and therefore left for future research.

Our plan for this paper is as follows. In Section 2, we provide all the notations used in the paper, state the whole list of required assumptions and define viscosity sub- and supersolutions along with equivalent characterizations. In Section 3 we construct two approximation schemes (an increasing and a decreasing on), consisting of sequences of penalized reflected BSDEs associated with standard switching problems. The counterpart of the decreasing scheme (resp. the increasing one) in terms of PDEs stands for the penalized scheme of system (1.1) (resp. system (1.2) given below). Section 4 is devoted to the proof of a comparison result related to the sub- and supersolutions of (1.1). In Section 5, the decreasing limit is identified as a viscosity subsolution of (1.1) while a super-solution is exhibited. Finally, we use Perron's method to construct a viscosity solution of (1.1) and, thanks to the uniqueness result, its connection with the limit of the decreasing scheme is obtained. As a by product, we also obtain a

similar characterization of the limit of the increasing scheme as the unique solution in viscosity sense of the following system of variational inequalities: For every $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$\begin{cases} \max\big\{\big(\underline{v}^{ij} - U^{ij}[\underline{\vec{v}}]\big)(t,x),\ \ \min\big\{\big(\underline{v}^{ij}(t,x) - L^{ij}[\underline{\vec{v}}]\big)(t,x), \\ \qquad -\partial_t \underline{v}^{ij}(t,x) - \mathcal{L}\underline{v}^{ij}(t,x) - f^{ij}(t,x,(\underline{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(t,x)D_x\underline{v}^{ij}(t,x))\big\}\big\} = 0, \\ \underline{v}^{ij}(T,x) = h^{ij}(x). \end{cases} \tag{1.2}$$

We do not know whether the solutions of (1.1) and (1.2) coincide or not. We note that this is a very important issue since this will enable to characterize this common solution as the value of the zero-sum switching game. Since it is beyond the scope of the paper, this question is left for further research.

2 Preliminaries and notation

Let T (resp. k, d) be a fixed positive constant (resp. two integers) and Γ^1 (resp. Γ^2) denote the set of switching modes for player 1 (resp. 2). For later use, we shall denote by Λ the cardinal of the product set $\Gamma^1 \times \Gamma^2$ and for $(i,j) \in \Gamma^1 \times \Gamma^2$, $(\Gamma^1)^{-i} := \Gamma^1 - \{i\}$ and $(\Gamma^2)^{-j} := \Gamma^2 - \{j\}$. Next, for $\vec{y} = (y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2} \in \mathbb{R}^\Lambda$, $(i,j) \in \Gamma^1 \times \Gamma^2$, and $\underline{y} \in \mathbb{R}$, we denote by $[\vec{y}^{-(ij)}, \underline{y}]$ the matrix which is obtained from $\vec{y}$ by replacing the element y^{ij} with $\underline{y}$.

Next, let us introduce the following functions. For any $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$\begin{aligned} &b:\ (t,x) \in [0,T] \times \mathbb{R}^k \mapsto b(t,x) \in \mathbb{R}^k; \\ &\sigma:\ (t,x) \in [0,T] \times \mathbb{R}^k \mapsto \sigma(t,x) \in \mathbb{R}^{k\times d}; \\ &f^{ij}:\ (t,x,\vec{y},z) \in [0,T] \times \mathbb{R}^{k+\Lambda+d} \mapsto f^{ij}(t,x,\vec{y},z)\ \in \mathbb{R}; \\ &\underline{g}_{ik}:\ (t,x) \in [0,T] \times \mathbb{R}^k \mapsto \underline{g}_{ik}(t,x) \in \mathbb{R} \qquad (k \in (\Gamma^1)^{-i}); \\ &\bar{g}_{jl}:\ (t,x) \in [0,T] \times \mathbb{R}^k \mapsto \bar{g}_{jl}(t,x) \in \mathbb{R} \qquad (l \in (\Gamma^2)^{-j}); \\ &h^{ij}:\ x \in \mathbb{R}^k \mapsto h^{ij}(x) \in \mathbb{R}. \end{aligned}$$

A function $\Phi:\ (t,x) \in [0,T] \times \mathbb{R}^k \mapsto \Phi(t,x) \in \mathbb{R}$ is called of *polynomial growth* if there exist two non-negative real constants C and γ such that

$$|\Phi(t,x)| \le C(1+|x|^\gamma), \quad (t,x) \in [0,T] \times \mathbb{R}^k.$$

Hereafter, this class of functions is denoted by Π_g. Let $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^k)$ (or simply $\mathcal{C}^{1,2}$) denote the set of real-valued functions defined on $[0,T]\times\mathbb{R}^k$, which are once (resp. twice) differentiable w.r.t. t (resp. x) and with continuous derivatives.

In this paper, we investigate existence and uniqueness of viscosity solutions $\vec{v}(t,x) := (v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}$ of the following system of variational inequalities with upper and lower interconnected obstacles: For any $(i,j) \in \Gamma^1\times\Gamma^2$,

$$\begin{cases} \min\big\{\big(v^{ij} - L^{ij}[\vec{v}]\big)(t,x),\ \ \max\big\{\big(v^{ij} - U^{ij}[\vec{v}]\big)(t,x), \\ \qquad -\partial_t v^{ij}(t,x) - \mathcal{L}v^{ij}(t,x) - f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(t,x)D_x v^{ij}(t,x))\big\}\big\} = 0 \\ v^{ij}(T,x) = h^{ij}(x) \end{cases} \tag{2.1}$$

where, for any $(t,x) \in [0,T]\times\mathbb{R}^k$,

$$\mathcal{L}\varphi(t,x) := b(t,x)D_x\varphi(t,x) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(t,x)D^2_{xx}\varphi(t,x)],$$

and $(i,j) \in \Gamma^1\times\Gamma^2$,

$$L^{ij}[\vec{v}](t,x) := \max_{k\in(\Gamma^1)^{-i}}(v^{kj}(t,x) - \underline{g}_{ik}(t,x)) \ \text{ and } \ U^{ij}[\vec{v}](t,x) = \min_{l\in(\Gamma^2)^{-j}}(v^{il}(t,x) + \bar{g}_{jl}(t,x)).$$

The functions f^{ij} stand for the instantaneous payoff when the first player is in mode i and the second one in mode j, and $\underline{g}_{ik}$ (resp. $\bar{g}_{jl}$) stands for the switching cost incurred when the first (resp. second) player decides to switch from mode i to mode k (resp. from mode j to mode l).

The lower obstacle $L^{ij}[\vec{v}]$ and an upper obstacle $U^{ij}[\vec{v}]$ are called interconnected because each of them depends on the underlying solution $\vec{v} := (v^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$.

In a way, the system (2.1) is the Bellman-Isaacs system of equations associated with the zero-sum switching game with utility functions $(f^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$, terminal payoffs $(h^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ and switching costs for the maximizer (resp. minimizer) given by $(\underline{g}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ (resp. $(\bar{g}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$)).

The following assumptions are in force throughout the rest of the paper.

(**H0**) The functions b and σ associated with the second order operator $\mathcal{L}$ are jointly continuous in (t,x), of linear growth in (t,x) and Lipschitz continuous w.r.t. x, meaning that there exists a non-negative constant C such that for any $(t,x,x')\in[0,T]\times\mathbb{R}^{k+k}$ we have:

$$|b(t,x)|+|\sigma(t,x)|\leq C(1+|x|) \text{ and } |\sigma(t,x)-\sigma(t,x')|+|b(t,x)-b(t,x')|\leq C|x-x'|.$$

(**H1**) Each function f^{ij}

(i) is continuous in (t,x) uniformly w.r.t. the other variables $(\vec{y},z)$ and for any (t,x) and the mapping $(t,x)\to f^{ij}(t,x,0,0)$ is of polynomial growth.

(ii) satisfies the standard hypothesis of Lipschitz continuity with respect to the variables $(\vec{y}:=(y^{ij})_{(i,j)\in\Gamma_1\times\Gamma_2},z)$, i.e. $\forall\ (t,x)\in[0,T]\times\mathbb{R}^k$, $\forall\ (\vec{y}_1,\vec{y}_2)\in\mathbb{R}^\Lambda\times\mathbb{R}^\Lambda$, $(z^1,z^2)\in\mathbb{R}^d\times\mathbb{R}^d$,

$$|f^{ij}(t,x,\vec{y}_1,z_1)-f^{ij}(t,x,\vec{y}_2,z_2)|\leq C\left(|\vec{y}_1-\vec{y}_2|+|z_1-z_2|\right),$$

where, $|\vec{y}|$ stands for the standard Euclidean norm of $\vec{y}$ in $\mathbb{R}^\Lambda$.

(**H2**) Monotonicity: Let $\vec{y}=(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$, then for any $(i,j)\in\Gamma^1\times\Gamma^2$ and any $(k,l)\neq(i,j)$ the mapping $y^{k,l}\to f^{ij}(s,\vec{y},z)$ is non-decreasing.

(**H3**) The functions h^{ij}, which are the terminal conditions in the system (2.1), are continuous with respect to x, belong to class Π_g and satisfy

$$\forall\ (i,j)\in\Gamma^1\times\Gamma^2 \text{ and } x\in\mathbb{R}^k,\ \max_{k\in(\Gamma^1)^{-i}}\big(h^{kj}(x)-\underline{g}_{ik}(T,x)\big)\leq h^{ij}(x)\leq\min_{l\in(\Gamma^2)^{-j}}\big(h^{il}(x)+\bar{g}_{jl}(T,x)\big).$$

(**H4**) The no free loop property: The switching costs $\underline{g}_{ik}$ and $\bar{g}_{jl}$ are non-negative, jointly continuous in (t,x), belong to Π_g and satisfy the following condition:

For any loop in $\Gamma^1\times\Gamma^2$, i.e., any sequence of pairs $(i_1,j_1),\ldots,(i_N,j_N)$ of $\Gamma^1\times\Gamma^2$ such that $(i_N,j_N)=(i_1,j_1)$, $\mathrm{card}\{(i_1,j_1),\ldots,(i_N,j_N)\}=N-1$ and $\forall\ q=1,\ldots,N-1$, either $i_{q+1}=i_q$ or $j_{q+1}=j_q$, we have $\forall (t,x)\in[0,T]\times\mathbb{R}^k$,

$$\sum_{q=1,N-1}\varphi_{i_qi_{q+1}}(t,x)\neq 0, \tag{2.2}$$

where, $\forall\ q=1,\ldots,N-1$, $\varphi_{i_qi_{q+1}}(t,x)=-\underline{g}_{i_qi_{q+1}}(t,x)1\!\!1_{i_q\neq i_{q+1}}+\bar{g}_{j_qi_{q+1}}(t,x)1\!\!1_{j_q\neq j_{q+1}}$ (resp. $\varphi_{i_qi_{q+1}}(t,x)=\underline{g}_{i_q,i_{q+1}}(t,x)1\!\!1_{i_q\neq i_{q+1}}-\bar{g}_{j_q,i_{q+1}}(t,x)1\!\!1_{j_q\neq j_{q+1}}$).

This assumption implies in particular that

$$\forall\,(i_1,\ldots,i_N)\in(\Gamma^1)^N \text{ such that } \; i_N=i_1 \text{ and } \operatorname{card}\{i_1,\ldots,i_N\}=N-1,\; \sum_{p=1}^{N-1}\underline{g}_{i_k,i_{k+1}}>0 \tag{2.3}$$

and

$$\forall\,(j_1,\ldots,j_N)\in(\Gamma^2)^N \text{ such that } j_N=j_1 \text{ and } \operatorname{card}\{j_1,\ldots,j_N\}=N-1,\; \sum_{p=1}^{N-1}\bar{g}_{j_k,j_{k+1}}>0. \tag{2.4}$$

By convention we set $\bar{g}_{j,j}=\underline{g}_{i,i}=0$.

Conditions (2.3) and (2.4) are classical in the literature of switching problems and usually referred to as the *no free loop property.*

Finally, let us mention that if we set

$$\underline{g}_{ij}(t,x)=|i-j|\underline{g}(t,x) \;\text{ and }\; \bar{g}_{ij}(t,x)=|i-j|\bar{g}(t,x),\quad (i,j)\in\Gamma^1\times\Gamma^2,$$

where both $\underline{g}$ and $\bar{g}$ are functions such that, for any $(t,x)\in[0,T]\times\mathbb{R}^k$, $\frac{\bar{g}(t,x)}{\underline{g}(t,x)}$ is not a rational number, then assumption (2.2) holds.

We now define the notions of viscosity super (or sub)-solution of the system (2.1). This is done in terms of the notions of *subjet* and *superjet* which we recall here.

Definition 1. *(Subjet and superjet)*
(i) For a lower semicontinuous (lsc) (resp. upper semicontinuous (usc)) function $u:[0,T]\times\mathbb{R}^k\to\mathbb{R}$, *we denote by* $J^-u(t,x)$ *(resp.* $J^+u(t,x)$*) the parabolic subjet (resp. superjet) of* u *at* $(t,x)\in[0,T]\times\mathbb{R}^k$, *as the set of triples* $(p,q,M)\in\mathbb{R}\times\mathbb{R}^k\times\mathbb{S}^k$ *satisfying*

$$\begin{aligned} u(t',x')\geq\; (\textit{resp.} \leq)\; & u(t,x)+p(t'-t)+q^\top(x'-x)+\\ & \tfrac{1}{2}(x'-x)^\top M(x'-x)+o\big(|t'-t|+|x'-x|^2\big) \end{aligned}$$

where $\mathbb{S}^k$ *is the set of symmetric real matrices of dimension* k.

(ii) For a lsc (rep. usc) function $u:[0,T]\times\mathbb{R}^k\to\mathbb{R}$, *we denote by* $\bar{J}^-u(t,x)$ *(resp.* $\bar{J}^+u(t,x)$*) the parabolic limiting subjet (resp. superjet) of* u *at* $(t,x)\in[0,T]\times\mathbb{R}^k$, *as the set of triples* $(p,q,M)\in\mathbb{R}\times\mathbb{R}^k\times\mathbb{S}^k$ *such that:*

$$\begin{aligned} &(p,q,M)=\lim_n(p_n,q_n,M_n),\; (t,x)=\lim_n(t_n,x_n) \textit{ with } (p_n,q_n,M_n)\in J^-u(t_n,x_n)\\ &(\textit{resp. } J^+u(t_n,x_n)) \textit{ and } u(t,x)=\lim_n u(t_n,x_n). \end{aligned}$$

Finally, given a locally bounded $\mathbb{R}$-valued deterministic function u, we denote by u_* (resp. u^*) its lower (resp. upper) semicontinuous envelope defined by:

$$\forall\,(t,x)\in[0,T]\times\mathbb{R}^k,\quad u_*(t,x)=\varliminf_{(t',x')\to(t,x);\;t'<T}u(t',x') \;\text{ and }\; u^*(t,x)=\varlimsup_{(t',x')\to(t,x);\;t'<T}u(t',x'). \tag{2.5}$$

We now give the definition of a viscosity solution for the system (2.1).

Definition 2. *Viscosity solution to (2.1)*
(i) A function $\vec{v}=(v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}:[0,T]\times\mathbb{R}^k\mapsto\mathbb{R}^\Lambda$ *such that for any* $(i,j)\in\Gamma^1\times\Gamma^2$, v^{ij} *is lsc (resp.*

usc), is called a viscosity supersolution (resp. a viscosity subsolution) to (2.1) if for any $(i,j) \in \Gamma^1 \times \Gamma^2$, for any $(t,x) \in [0,T) \times \mathbb{R}^k$ and any $(p,q,M) \in \bar{J}^- v^{ij}(t,x)$ (resp. $\bar{J}^+ v^{ij}(t,x)$) we have:

$$\begin{cases} \min\big\{v^{ij}(t,x) - L^{ij}[\vec{v}](t,x), \\ \quad \max\big\{-p - b(t,x).q - \frac{1}{2}Tr[(\sigma\sigma^\top)(t,x)M] - f^{ij}(t,x,\vec{v}(t,x),\sigma^\top(t,x)q); v^{ij}(t,x) - U^{ij}[\vec{v}](t,x)\big\}\big\} \geq 0, \\ v_i(T,x) \geq \ h^{ij}(x), \end{cases} \tag{2.6}$$

$\Big($*resp.*

$$\begin{cases} \min\big\{v^{ij}(t,x) - L^{ij}[\vec{v}](t,x), \\ \quad \max\big\{-p - b(t,x).q - \frac{1}{2}Tr[(\sigma\sigma^\top)(t,x)M] - f^{ij}(t,x,\vec{v}(t,x),\sigma^\top(t,x)q); v^{ij}(t,x) - U^{ij}[\vec{v}](t,x)\big\}\big\} \leq 0, \\ v^{ij}(T,x) \leq \ h^{ij}(x)\Big). \end{cases} \tag{2.7}$$

(ii) A locally bounded function $\vec{v} = (v^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2} : [0,T] \times \mathbb{R}^k \to \mathbb{R}^\Lambda$ is called a viscosity solution of (2.1) if the associated lower (resp. upper) semicontinuous envelope $(v_*^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$ (resp. $(v^{kl^*})_{(k,l)\in\Gamma^1\times\Gamma^2}$) defined in (2.5) is a viscosity supersolution (resp. subsolution) of (2.1).
If, in addition, for any $(k,l) \in \Gamma^1 \times \Gamma^2$, $v_*^{kl} = v^{kl*}$, then $(v^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$ is a continuous viscosity solution of (2.1).

Remark 1. *Under Assumptions (H0)-(H4), the above definition of a viscosity solution for (2.1) can be relaxed replacing the limiting subjet $\bar{J}^-(v_*^{ij})(t,x)$ of the supersolution v_* (resp. the limiting superjet $\bar{J}^+(v^{ij,*})(t,x)$ of v^*) by the subjet $J^-(v_*^{ij})(t,x)$ (resp. by the superjet $J^+(v^{ij,*})(t,x)$). This results from the continuity of the functions b, σ, $(f^{ij}, h^{ij}, \underline{g}_{ij}, \bar{g}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ and the monotonicity property (H2) of $(f^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$.*

3 Systems of reflected BSDEs and approximation schemes of the solutions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space on which is defined a standard d-dimensional Brownian motion $B = (B_t)_{0\leq t\leq T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{0\leq t\leq T}$. Let $\mathbf{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ be the completed filtration of $(\mathcal{F}_t^0)_{0\leq t\leq T}$ with the $\mathbb{P}$-null sets of $\mathcal{F}$, hence $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the usual conditions, i.e., it is right continuous and complete. Furthermore, let

- $\mathcal{P}$ be the σ-algebra on $[0,T] \times \Omega$ of $\mathbf{F}$-progressively measurable sets;
- $\mathcal{H}^{2,\ell}$ ($\ell \geq 1$) be the set of $\mathcal{P}$-measurable and $\mathbb{R}^\ell$-valued processes $w = (w_t)_{t\leq T}$ such that $\mathbb{E}[\int_0^T |w_s|^2 ds] < \infty$;
- $\mathcal{S}^{2,\ell}$ ($\ell \geq 1$) be the subset of $\mathcal{H}^{2,\ell}$ of continuous processes such that $\mathbb{E}[\sup_{t\leq T} |w_t|^2] < \infty$. Finally let $\mathcal{A}^{+,2}$ be the subset of $\mathcal{S}^{2,1}$ of non-decreasing processes $K = (K_t)_{t\leq T}$ such that $K_0 = 0$ and $\mathbb{E}[K_T^2] < \infty$.

Next, for $n, m \geq 0$, let $(Y^{ij,n,m}, Z^{ij,n,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ be the solution of the following system of BSDEs.

$$\begin{cases} (Y^{ij,n,m}, Z^{ij,n,m}) \in \mathcal{S}^{2,1} \times \mathcal{H}^{2,d}; \\ dY_s^{ij,n,m} \ = \ -f^{ij,n,m}(s, X_s^{t,x}, (Y_s^{kl,n,m})_{(k,l)\in\Gamma^1\times\Gamma^2}, Z_s^{ij,n,m})ds + Z_s^{ij,n,m} dB_s, \ \ s \leq T, \\ Y_T^{ij,nm} = h^{ij}(X_T^{t,x}), \end{cases} \tag{3.1}$$

where,

$$\begin{aligned} &f^{ij,n,m}(s, X_s^{t,x}, (y^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}, z) := f^{ij}(s, X_s^{t,x}, (y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}, z) + \\ &\qquad n(y^{ij} - \max_{k\in(\Gamma^1)^{-i}}\{y^{kj} - \underline{g}_{ik}(s, X_s^{t,x})\})^- - m(y^{ij} - \min_{l\in(\Gamma^2)^{-j}}\{y^{il} + \bar{g}_{jl}(s, X_s^{t,x})\})^+. \end{aligned}$$

Note that by Assumption (H1) and the standard result on multi-dimensional BSDEs by Pardoux and Peng (1990) ([24]), the solution $(Y^{ij,n,m}, Z^{ij,n,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ exists and is unique. Since the generators and the terminal values depend on (t,x), the processes $Y^{ij,n,m}$ and $Z^{ij,n,m}$ also depend on (t,x) but, to avoid overload notation, we do not mention this dependence in the sequel. Furthermore, the following monotonicity properties holds for the double sequence $(Y^{ij,n,m})_{n,m}$.

Proposition 3.1. *For any $(i,j)\in\Gamma^1\times\Gamma^2$ and $n,m\ge 0$ we have*

$$\mathbb{P}-a.s., \quad Y^{ij,n,m}\le Y^{ij,n+1,m} \ \text{ and } \ Y^{ij,n,m}\ge Y^{ij,n,m+1}, \qquad (i,j)\in\Gamma^1\times\Gamma^2. \tag{3.2}$$

Moreover, for any $(i,j)\in\Gamma^1\times\Gamma^2$ and $n,m\ge 0$, there exists a deterministic continuous function $v^{ij,n,m}$ in Π_g such that, for any $t\le T$,

$$Y_s^{ij,n,m}=v^{ij,n,m}(s,X_s^{t,x}), \quad s\in[t,T]. \tag{3.3}$$

Finally, for any $(i,j)\in\Gamma^1\times\Gamma^2$ and $n,m\ge 0$,

$$v^{ij,n,m}(t,x)\le v^{ij,n+1,m}(t,x) \ \text{ and } \ v^{ij,n,m}(t,x)\ge v^{ij,n,m+1}(t,x), \qquad (t,x)\in[0,T]\times\mathbb{R}^k. \tag{3.4}$$

Proof. The second claim is just the representation of solutions of standard BSDEs by deterministic functions in the Markovian framework (see e.g. El Karoui *et al.* (1997) ([13]) for more details). As for the first one, it is based on the result by Hu and Peng (2006) ([22]) related to the comparison of solutions of multi-dimensional BSDEs (we recall in Appendix (A1)). To prove this, it is enough to show that for any t, $(y_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$, $(\bar{y}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}\in\mathbb{R}^\Lambda$ and $(z_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$, $(\bar{z}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}\in(\mathbb{R}^d)^\Lambda$ we have:

$$\begin{aligned}-4\textstyle\sum_{(i,j)\in\Gamma^1\times\Gamma^2} y_{ij}^-\left(f^{ij,n,m}(s,X_s^{t,x},(y_{kl}^++\bar{y}_{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z_{ij})-f^{ij,n+1,m}(s,X_s^{t,x},(\bar{y}_{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},\bar{z}_{ij})\right)\\ \le 2\textstyle\sum_{(i,j)\in\Gamma^1\times\Gamma^2}1_{[y_{ij}<0]}\|z_{ij}-\bar{z}_{ij}\|^2+C\sum_{(i,j)\in\Gamma^1\times\Gamma^2}(y_{ij}^-)^2,\end{aligned}$$

where, C is a constant, $y_{ij}^-=\max(-y_{ij},0)$ and $y_{ij}^+=\max(y_{ij},0)$. This inequality follows easily from the fact that, for any $(i,j)\in\Gamma^1\times\Gamma^2$,

(i) $f^{ij,n,m}(s,X_s^{t,x},(y_{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z_{ij})\le f^{ij,n+1,m}(s,X_s^{t,x},(\bar{y}_{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z_{ij})$,

(ii) For any $(u_{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}\in\mathbb{R}^\Lambda$ such that $u_{ij}=0$ and $u_{kl}\ge 0$, for $(k,l)\ne(i,j)$,

$$f^{ij,n,m}(s,X_s^{t,x},(y_{kl}+u_{kl})_{(i,j)\in\Gamma^1\times\Gamma^2},z_{ij})\ge f^{ij,n,m}(s,X_s^{t,x},(y_{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z_{ij}).$$

(iii) f_{ij} depends only on z_{ij} and not on the other components z_{kl}, $(k,l)\ne(i,j)$.

Consequently, we have

$$\mathbb{P}-a.s., \qquad Y^{ij,n,m}\le Y^{ij,n+1,m}, \qquad (i,j)\in\Gamma^1\times\Gamma^2.$$

In the same way one can show that $Y^{ij,n,m+1}\le Y^{ij,n,m}$. Finally, the inequalities of (3.4) are obtained by taking $s=t$ in (3.2) in view of the representation (3.3) of $Y^{ij,n,m}$ by $v^{ij,n,m}$ and $X^{t,x}$. ■

We now suggest two approximation schemes obtained from the sequence $(Y^{ij,n,m},\ (i,j)\in\Gamma^1\times\Gamma^2)_{n,m}$ of the solution of the system (3.1). The first scheme is a sequence of decreasing reflected BSDEs with interconnected lower obstacles and the second one is an increasing sequence of reflected BSDEs with interconnected upper obstacles.

Let us first introduce the decreasing approximation scheme by considering the following system of reflected BSDEs with interconnected obstacles: $\forall(i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases}\bar{Y}^{ij,m}\in\mathcal{S}^{2,1},\ \bar{Z}^{ij,m}\in\mathcal{H}^{2,d} \text{ and } \bar{K}^{ij,m}\in\mathcal{A}^{2,+}\ ;\\ \bar{Y}_s^{ij,m}=h^{ij}(X_T^{t,x})+\int_s^T\bar{f}^{ij,m}(r,X_r^{t,x},(\bar{Y}_r^{kl,m})_{(k,l)\in\Gamma^1\times\Gamma^2},\bar{Z}_r^{ij,m})dr+\bar{K}_T^{ij,m}-\bar{K}_s^{ij,m}-\int_s^T\bar{Z}_r^{ij,m}dB_r,\\ \bar{Y}_s^{ij,m}\ge\max\limits_{k\in(\Gamma^1)^{-i}}\{\bar{Y}_s^{kj,m}-\underline{g}_{ik}(s,X_s^{t,x})\}, \qquad s\le T,\\ \int_0^T(\bar{Y}_s^{ij,m}-\max_{k\in(\Gamma^1)^{-i}}\{\bar{Y}_s^{kj,m}-\underline{g}_{ik}(s,X_s^{t,x})\})d\bar{K}_s^{ij,m}=0,\end{cases} \tag{3.5}$$

where, for any $(i,j)\in\Gamma^1\times\Gamma^2$, $m\geq 0$ and $(s,\vec{y},z^{ij})$,

$$\bar{f}^{ij,m}(s,X_s^{t,x},\vec{y},z^{ij}):=f^{ij}(s,X_s^{t,x},(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z^{ij})-m\big(y^{ij}-\min_{l\in(\Gamma^2)^{-j}}(y^{il}+\bar{g}_{jl}(s,X_s^{t,x}))\big)^+.$$

Thanks to Assumptions (H1)-(H3) and (2.3), by Theorem 3.5 in Hamadène and Zhang (2010) ([17]), the solution of (3.5) exists and is unique. In fact, this holds again under weaker assumptions (see Hamadène and Morlais (2011)([19], Theorem 1). For sake of completeness, a statement of this recent result is given in Appendix (A2). Moreover, we have the following properties.

Proposition 3.2. *For any $(i,j)\in\Gamma^1\times\Gamma^2$ and $m\geq 0$, we have*
(i)

$$\mathbb{E}[\sup_{t\leq s\leq T}|Y_s^{ij,n,m}-\bar{Y}_s^{ij,m}|^2]\rightarrow 0,\quad as\quad n\rightarrow\infty, \tag{3.6}$$

(ii)

$$\mathbb{P}-a.s.,\qquad \bar{Y}^{ij,m}\geq\bar{Y}^{ij,m+1}.$$

(iii) There exists a unique Λ-uplet of deterministic continuous functions $(\bar{v}^{kl,m})_{(k,l)\in\Gamma^1\times\Gamma^2}$ in Π_g such that, for every $t\leq T$,

$$\bar{Y}_s^{ij,m}=\bar{v}^{ij,m}(s,X_s^{t,x}),\qquad s\in[t,T]. \tag{3.7}$$

Moreover, for any $(i,j)\in\Gamma^1\times\Gamma^2$ and $(t,x)\in[0,T]\times\mathbb{R}^k$, $\bar{v}^{ij,m}(t,x)\geq\bar{v}^{ij,m+1}(t,x)$.

Finally, $(\bar{v}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the unique viscosity solution in the class Π_g of the following system of variational inequalities with inter-connected obstacles. $\forall\ (i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases}\min\{\bar{v}^{ij,m}(t,x)-\max\limits_{k\in(\Gamma^1)^{-i}}(\bar{v}^{kj,m}(t,x)-\underline{g}_{ik}(t,x));\\ \qquad -\partial_t\bar{v}^{ij,m}(t,x)-b(t,x)D_x\bar{v}^{ij,m}(t,x)-\frac{1}{2}Tr(\sigma\sigma^\top(t,x)D_{xx}^2\bar{v}^{ij,m}(t,x))\\ \qquad\qquad -\bar{f}^{ij,m}(t,x,(\bar{v}^{kl,m}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x)D_x\bar{v}^{ij,m}(t,x))\}=0,\\ \bar{v}^{ij,m}(T,x)=h^{ij}(x)\end{cases} \tag{3.8}$$

where,

$$\bar{f}^{ij,m}(s,x,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z)=f^{ij}(s,x,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z)-m\big(y^{ij}-\min_{l\in(\Gamma^2)^{-j}}(y^{il}+\bar{g}_{jl}(s,x))\big)^+.$$

Proof. Let us prove (i). For this, it is enough to consider the case $m=0$, and we will do so, since for any $(i,j)\in\Gamma^1\times\Gamma^2$, the function

$$(s,x,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2})\longrightarrow -m\big(y^{ij}-\min_{l\in(\Gamma^2)^{-j}}(y^{il}+\bar{g}_{jl}(s,x))\big)^+$$

has the same properties as f^{ij} displayed in (H1)-(H2).

To begin with, let us show that for any i,j and $n\geq 0$,

$$\mathbb{P}-a.s.,\qquad Y^{ij,n,0}\leq\bar{Y}^{ij,0}. \tag{3.9}$$

First andt w.l.o.g. we may assume that f^{ij} is non-decreasing w.r.t. $(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$, since thanks to Assumption (H2), it is enough to multiply the solutions by $e^{\varpi t}$, where ϖ is appropriately chosen in order to fall in this latter case, since f^{ij} is Lipschitz in y^{ij}. Now, for fixed n, let us define recursively the sequence $(\tilde{Y}^{k,ij,n})_{k\geq 0}$ as follows:

For $k=0$ and any $(i,j)\in\Gamma^1\times\Gamma^2$, we set $\tilde Y^{0,ij,n}:=\bar Y^{ij,0}$ and, for any $k\geq 1$, let us define $(\tilde Y^{k,ij,n},\tilde Z^{k,ij,n})\in\mathcal{S}^{2,1}\times\mathcal{H}^{2,d}$ as the solution of the following system of BSDEs: $\forall(i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases} -d\tilde Y_s^{k,ij,n} \;=\; f^{ij}(s,X_s^{t,x},(\tilde Y_s^{k-1,pq,n})_{(p,q)\in\Gamma^1\times\Gamma^2},\tilde Z_s^{k,ij,n})ds+ \\ \qquad\qquad n\{\tilde Y_s^{k,ij,n}-\max_{l\in(\Gamma^1)^{-i}}(\tilde Y_s^{k-1,lj,n}-\underline{g}_{ik}(s,X_s^{t,x}))\}^-ds-\tilde Z_s^{k,ij,n}dB_s, \\ \tilde Y_T^{k,ij,n}=h^{ij}(X_T^{t,x}) \end{cases} \tag{3.10}$$

The solution of (3.10) exists since it is a multi-dimensional standard BSDE with a Lipschitz coefficient, noting that $(\tilde Y_s^{k-1,pq,n})_{(p,q)\in\Gamma^1\times\Gamma^2}$ is already given. Since, n is fixed and the coefficient

$$\varphi^{ij,n}(s,\omega,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z^{ij}):=f^{ij}(s,X_s^{t,x},(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z^{ij})+n\{y^{ij}-\max_{l\in(\Gamma^1)^{-i}}(y^{lj}-\underline{g}_{il}(s,X_s^{t,x})\}^-$$

is Lipschitz w.r.t. $((y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z^{ij})$, the sequence $(\tilde Y^{k,ij,n})_k$ converges in $\mathcal{S}^{2,1}$ to $Y^{ij,n,0}$, as $k\to\infty$, for any i,j and n.
Using an induction argument w.r.t. k, we prove that, for any i,j and n,

$$\mathbb{P}-a.s.,\qquad \tilde Y^{k,ij,n}\leq\bar Y^{ij,0},\qquad k\geq 0.$$

Indeed, for $k=0$ the inequalities hold true through the definition of $\tilde Y^{0,ij,n}$. Assume now that these inequalities are valid for some $k-1$, i.e., for any i,j and n,

$$\mathbb{P}-a.s.,\qquad \tilde Y^{k-1,ij,n}\leq\bar Y^{ij,0}. \tag{3.11}$$

Thus, taking into account that $\bar Y^{ij,0}$ satisfies (3.5) with $m=0$ and the fact that f^{ij} is non-decreasing w.r.t. $(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$ then for any i,j,n, it holds

$$\begin{aligned} &f^{ij}(s,X_s^{t,x},(\tilde Y_s^{k-1,pq,n})_{(p,q)\in\Gamma^1\times\Gamma^2},z^{ij})+n\{\bar Y^{ij,0}-\max_{l\in(\Gamma^1)^{-i}}(\tilde Y_s^{k-1,lj,n}-\underline{g}_{il}(s,X_s^{t,x}))\}^- \\ &\quad\leq f^{ij}(s,X_s^{t,x},(\bar Y_s^{pq,0})_{(p,q)\in\Gamma^1\times\Gamma^2},z^{ij}). \end{aligned}$$

Using the standard comparison result of solutions of one dimensional BSDEs we obtain that, for any n,i,j, $\tilde Y^{k,ij,n}\leq\bar Y^{ij,0}$. a.s. Thus the property (3.11) is valid for any k. Taking the limit as k tends to ∞, we obtain (3.9).

We can now apply Peng's monotonic limit (see Peng (1999) ([26])) to the increasing sequence $(Y^{ij,n,0})_{n\geq 0}$. This yields the existence of processes $\hat Y^{ij}$, $\hat Z^{ij}$ and $\hat K^{ij}$, $(i,j)\in\Gamma^1\times\Gamma^2$, such that:
(a) $\hat Y^{ij}$ is RCLL and uniformly $\mathbb{P}$-square integrable. Moreover, for any stopping time τ, $\lim_{n\to\infty}\nearrow Y_\tau^{ij,n,0}=\hat Y_\tau^{ij}$.
(b) $\hat K^{ij}$ is RCLL non-decreasing, $\hat K_0^{ij}=0$ and for any stopping time τ, $\lim_{n\to\infty}K_\tau^{ij,n,0}=\hat K_\tau^{ij}$, $\mathbb{P}-a.s.$
(c) $\hat Z^{ij}\in\mathcal{H}^{2,d}$ and for any $p\in[1,2)$,

$$\lim_{n\to\infty}\mathbb{E}[\int_0^T|Z_s^{ij,n,0}-\hat Z_s^{ij}|^p ds]=0.$$

(d) For any $(i,j)\in\Gamma^1\times\Gamma^2$, the triple $(\hat Y^{ij},\hat Z^{ij},\hat K^{ij})$ satisfies: $\forall s\leq T$,

$$\begin{cases} \hat Y_s^{ij}=h^{ij}(X_T^{t,x})+\int_s^T f^{ij}(r,X_r^{t,x},(\hat Y_r^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},\hat Z_r^{ij})dr+\hat K_T^{ij}-\hat K_s^{ij}-\int_s^T\hat Z_r^{ij}dB_r \\ \hat Y_s^{ij}\geq\max_{k\in(\Gamma^1)^{-i}}\{\hat Y_s^{kj}-\underline{g}_{ik}(s,X_s^{t,x})\}. \end{cases} \tag{3.12}$$

The remaining of the proof is the same as in Hamadène and Zhang (2010) (([17]), Theorem 3.2) and it mainly consists in proving both the continuity of $\hat Y^{ij}$ and $\hat K^{ij}$ and the Skorohod condition (these properties are deduced using the no-free loop property (H4)). Finally we obtain that the triple $(\hat Y^{ij},\hat Z^{ij},\hat K^{ij})$ satisfies (3.5) with $m=0$. Hence, by uniqueness of the solutions of (3.5), $\hat Y^{ij}=\bar Y^{ij,0}$ a.s, which completes the proof of (i).

(ii) is an immediate consequence of (i) and Proposition 3.1.

We now establish (iii). The existence of the deterministic continuous functions $(\bar{v}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ such that for any $s\in[t,T]$, $Y_s^{ij,m}=\bar{v}^{ij,m}(s,X_s)$, $(i,j)\in\Gamma^1\times\Gamma^2$, $m\geq 0$, and which satisfy the system (3.8), is obtained in [19] (see Appendix A2, Theorem 6.2). Finally as $Y^{ij,m}\geq Y^{ij,m+1}$ a.s., we deduce that $\bar{v}^{ij,m}\geq\bar{v}^{ij,m+1}$, taking into account (ii), which completes the proof. ■

We now introduce the increasing approximation scheme by considering the following system of reflected BSDEs with interconnected obstacles: for any $(i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases} \underline{Y}^{ij,n}\in\mathcal{S}^2,\ \underline{Z}^{ij,n}\in\mathcal{H}^{2,d} \text{ and } \underline{K}^{ij,n}\in\mathcal{A}^{2,+}\ ; \\ \underline{Y}_s^{ij,n}=h^{ij}(X_T^{t,x})+\int_s^T \underline{f}^{ij,n}(r,X_r^{t,x},(\underline{Y}_r^{kl,n})_{(k,l)\in\Gamma^1\times\Gamma^2},\underline{Z}_r^{ij,n})dr-(\underline{K}_T^{ij,n}-\underline{K}_s^{ij,n})-\int_s^T \underline{Z}_r^{ij,n}dB_r, \\ \underline{Y}_s^{ij,n}\leq \min\limits_{l\in(\Gamma^2)^{-j}}\{\underline{Y}_s^{il,n}+\bar{g}_{jl}(s,X_s^{t,x})\},\quad s\leq T, \\ \displaystyle\int_0^T(\underline{Y}_s^{ij,n}-\min_{l\in(\Gamma^2)^{-j}}\{\underline{Y}_s^{il,n}+\bar{g}_{jl}(s,X_s^{t,x})\})d\underline{K}_s^{ij,n}=0, \end{cases} \tag{3.13}$$

where, for any $(i,j)\in\Gamma^1\times\Gamma^2$, $n\geq 0$ and $(s,\vec{y},z^{ij})$,

$$\underline{f}^{ij,n}(s,X_s^{t,x},\vec{y},z^{ij})=f^{ij}(s,X_s^{t,x},(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z^{ij})+n\big(y^{ij}-\max_{k\in(\Gamma^1)^{-i}}\{y^{kj}-\underline{g}_{ik}(s,X_s^{t,x})\}\big)^-.$$

Thanks to assumptions (H1)-(H3) and (H4)-(2.4), by Theorems 3.2 and 3.5 in Hamadène and Zhang (2010) ([17]) the solution of (3.5) exists and is unique.

Below is the analogous of Proposition 3.2. We do not give its proof since it is deduced from this latter proposition in considering the equation satisfied by $(-\underline{Y}^{ij,n},-\underline{Z}^{ij,n},\underline{K}^{ij,n})$.

Proposition 3.3. *For any $(i,j)\in\Gamma^1\times\Gamma^2$ we have*
(i)

$$\mathbb{E}[\sup_{t\leq s\leq T}|Y_s^{ij,n,m}-\underline{Y}_s^{ij,n}|^2]\to 0\qquad as\qquad m\to\infty. \tag{3.14}$$

(ii) For any $n\geq 0$,

$$\mathbb{P}-a.s.,\qquad \underline{Y}^{ij,n}\leq\underline{Y}^{ij,n+1}.$$

(iii) For every $n\geq 0$, there exists a unique Λ-uplet of deterministic continuous functions $(\underline{v}^{kl,n})_{(k,l)\in\Gamma^1\times\Gamma^2}$ in Π_g such that, for every $t\leq T$,

$$\underline{Y}_s^{ij,n}=\underline{v}^{ij,n}(s,X_s^{t,x}),\qquad t\leq s\leq T. \tag{3.15}$$

Moreover, for any $n\geq 0$ and $(i,j)\in\Gamma^1\times\Gamma^2$,

$$\underline{v}^{ij,n}(t,x)\leq\underline{v}^{ij,n+1}(t,x),\qquad (t,x)\in[0,T]\times\mathbb{R}^k.$$

Finally, $(\underline{v}^{ij,n})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the unique viscosity solution in the class Π_g of the following system of variational inequalities with inter-connected obstacles: $\forall\,(i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases} \max\{\underline{v}^{ij,n}(t,x)-\min\limits_{l\in(\Gamma^2)^{-j}}(\underline{v}^{il,n}(t,x)+\bar{g}_{ik}(t,x)); \\ \qquad -\partial_t\underline{v}^{ij,n}(t,x)-b(t,x)D_x\underline{v}^{ij,n}(t,x)-\frac{1}{2}Tr(\sigma\sigma^\top(t,x)D_{xx}^2\underline{v}^{ij,n}(t,x)) \\ \qquad\qquad -\underline{f}^{\,ij,n}(t,x,(\underline{v}^{kl,n}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x)D_x\underline{v}^{ij,n}(t,x))\}=0, \\ \underline{v}^{ij,n}(T,x)=h^{ij}(x) \end{cases} \tag{3.16}$$

where,

$$\underline{f}^{\,ij,n}(s,x,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z)=f^{ij}(s,x,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z)+n\{y^{ij}-\max_{k\in(\Gamma^1)^{-i}}(y^{kj}-\underline{g}_{ik}(s,x))\}^-.$$

For $(t,x) \in [0,T] \times \mathbb{R}^k \in [0,T] \times \mathbb{R}^k$ and $(i,j) \in \Gamma^1 \times \Gamma^2$, let us define

$$\bar{v}^{ij}(t,x) := \lim_{m\to\infty} \bar{v}^{ij,m}(t,x) \quad \text{and} \quad \underline{v}^{ij}(t,x) := \lim_{n\to\infty} \underline{v}^{ij,n}(t,x).$$

Then, as a by-product of Propositions 3.2 and 3.3 we have

Corollary 1. *For any $(i,j) \in \Gamma^1 \times \Gamma^2$, the function $\bar{v}^{ij}$ (resp. $\underline{v}^{ij}$) is usc (resp. lsc). Moreover, $\bar{v}^{ij}$ and $\underline{v}^{ij}$ belong to Π_g i.e., there exist deterministic constants $C \geq 0$ and $\gamma > 0$ such that, for any $(t,x) \in [0,T] \times \mathbb{R}^k$,*

$$|\bar{v}^{ij}(t,x)| + |\underline{v}^{ij}(t,x)| \leq C(1+|x|^\gamma), \qquad (t,x) \in [0,T] \times \mathbb{R}^k.$$

Finally, for any $(t,x) \in [0,T] \times \mathbb{R}^k$,

$$\underline{v}^{ij}(t,x) \leq \bar{v}^{ij}(t,x).$$

Proof. For any $(i,j) \in \Gamma^1 \times \Gamma^2$, $\bar{v}^{ij}$ (resp. $\underline{v}^{ij}$) is obtained as a decreasing (resp. increasing) limit of continuous functions. Therefore, it is usc (resp. lsc). Next, for any (i,j) and n,m,

$$v^{ij,n,m}(t,x) \leq v^{ij,n,0}(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^k,$$

as the sequence $(v^{ij,n,m})_{m\geq 0}$ is decreasing. Thus, taking the limit as $m \to \infty$ we obtain

$$\underline{v}^{ij,n} \leq v^{ij,n,0}.$$

Now, using (3.3) and (3.6) it follows that, for any $t \leq T$ and $s \in [t,T]$, $Y_s^{ij,n,0} = v^{ij,n,0}(s, X_s^{t,x})$ and the processes $Y^{ij,n,0}$ converges in $\mathcal{S}^2$, as $n \to \infty$, to $\bar{Y}^{ij,0}$ solution of (3.5) with $m = 0$. Furthermore, by (3.7), there exists a deterministic continuous function $\bar{v}^{ij,0}$ of class Π_g such that for any $t \leq T$ and $s \in [t,T]$, $\bar{Y}_s^{ij,0} = \bar{v}^{ij,0}(s, X_s^{t,x})$. Then, taking $s = t$ and the limit as $n \to \infty$ to obtain

$$\underline{v}^{ij}(t,x) := \lim_{n\to\infty} \underline{v}^{ij,n}(t,x) \leq \lim_{n\to\infty} v^{ij,n,0}(t,x) = \bar{v}^{ij,0}(t,x), \ \forall (t,x) \in [0,T] \times \mathbb{R}^k.$$

But, $\bar{v}^{ij,0}$ and $\underline{v}^{ij,n}$ belong to Π_g and $\underline{v}^{ij,n} \leq \underline{v}^{ij,n+1}$. Thus, $\underline{v}^{ij}$ belongs to Π_g, for any $(i,j) \in \Gamma^1 \times \Gamma^2$. In the same way one can show that $\bar{v}^{ij}$ belongs to Π_g, for any $(i,j) \in \Gamma^1 \times \Gamma^2$. The last inequality follows from (3.4) and the definitions of $\bar{v}^{ij}$ and $\underline{v}^{ij}$. ■

4 A comparison result

In this section we investigate some qualitative properties of viscosity solutions of the system (2.1). In particular, we show in Corollary 2 below that if the system (2.1) admits a viscosity solution in the class Π_g, then it is unique and continuous.

We first show that if $(u^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ (resp. $(w^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$) is an *usc* subsolution (resp. *lsc* supersolution) of (2.1) which belongs to Π_g, then for any $(i,j) \in \Gamma^1 \times \Gamma^2$, $u^{ij} \leq w^{ij}$. To begin with, we give an intermediary result which is required in the second step of the proof of the comparison result.

Lemma 4.1. *Let $\vec{u} := (u^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ (resp. $\vec{w} := (w^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$) be an usc subsolution (resp. lsc supersolution) of (2.1). Let $(t,x) \in [0,T] \times \mathbb{R}^k$ and let $\tilde{\Gamma}(t,x)$ be the following set.*

$$\tilde{\Gamma}(t,x) := \{(i,j) \in \Gamma^1 \times \Gamma^2, u^{ij}(t,x) - w^{i,j}(t,x) = \max_{(k,l)\in\Gamma^1\times\Gamma^2} \{u^{kl}(t,x) - w^{kl}(t,x)\}.$$

Then, there exists $(i_0,j_0) \in \tilde{\Gamma}(t,x)$ such that

$$u^{i_0j_0}(t,x) > L^{i_0j_0}[\vec{u}](t,x) \quad \text{and} \quad w^{i_0j_0}(t,x) < U^{i_0j_0}[\vec{w}](t,x). \tag{4.1}$$

Proof: Let $(t,x) \in [0,T] \times \mathbb{R}^k$ be fixed. Since the set $\Gamma^1 \times \Gamma^2$ is finite then $\tilde{\Gamma}(t,x)$ is not empty. Next, let us show the claim by contradiction. So for any $(i,j) \in \tilde{\Gamma}(t,x)$ either $u^{ij}(t,x) \leq L^{ij}[\vec{u}](t,x)$ or $w^{ij}(t,x) \geq U^{ij}[\vec{w}](t,x)$ holds. Let us first assume that:

$$u^{ij}(t,x) \leq L^{ij}[\vec{u}](t,x). \tag{4.2}$$

Then, there exists some $k \in (\Gamma^1)^{-i}$, such that

$$u^{ij}(t,x) \leq L^{ij}[\vec{u}](t,x) = u^{kj}(t,x) - \underline{g}_{ik}(t,x).$$

But, since $\vec{w}$ is a supersolution to (2.1) we also deduce

$$w^{ij}(t,x) \geq w^{kj}(t,x) - \underline{g}_{ik}(t,x),$$

which implies that

$$u^{ij}(t,x) - u^{kj}(t,x) \leq -\underline{g}_{ik}(t,x) \leq w^{ij}(t,x) - w^{kj}(t,x)$$

and then

$$u^{ij}(t,x) - w^{ij}(t,x) \leq u^{kj}(t,x) - w^{kj}(t,x).$$

Since, by assumption $(i,j) \in \tilde{\Gamma}(t,x)$, the two previous inequalities are instead equalities, (k,j) belongs to $\tilde{\Gamma}(t,x)$, $k \neq i$ and finally it holds that

$$u^{ij}(t,x) - u^{kj}(t,x) = -\underline{g}_{ik}(t,x) = w^{ij}(t,x) - w^{kj}(t,x). \tag{4.3}$$

Next, if (4.2) does not hold, then necessarily $u^{ij}(t,x) > L^{ij}[\vec{u}](t,x)$ and $w^{ij}(t,x) \geq U^{ij}[\vec{w}](t,x)$. As $(u^{ij})_{i,j}$ is a subsolution of (2.1) we obtain

$$u^{ij}(t,x) \leq U^{ij}[\vec{u}](t,x) \leq u^{il}(t,x) + \bar{g}_{jl}(t,x), \quad l \in {\Gamma_2}^{-j}.$$

On the other hand, for some index $l \in {\Gamma_2}^{-j}$ it holds

$$w^{ij}(t,x) - w^{il}(t,x) \geq \bar{g}_{jl}(t,x) \geq u^{ij}(t,x) - u^{il}(t,x).$$

Once more as $(i,j) \in \tilde{\Gamma}(t,x)$ then the previous inequalities yield that $(i,l) \in \tilde{\Gamma}(t,x)$, $l \neq j$ and

$$u^{ij}(t,x) - u^{il}(t,x) = \bar{g}_{jl}(t,x) = w^{ij}(t,x) - w^{il}(t,x). \tag{4.4}$$

Repeating now this reasoning as many times as necessary, and since $\Gamma^1 \times \Gamma^2$ is finite, there exits a loop $(i_1,j_1),\dots,(i_{N-1},j_{N-1}),(i_N,j_N) = (i_1,j_1)$ such that

$$\sum_{q=1,N-1} \varphi_{i_q,i_{q+1}}(t,x) = 0,$$

which contradicts Assumption (H4), whence the claim is proved. ■

We now give the main result of this subsection.

Theorem 4.2. *Assume that $\vec{u} = (u^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ (resp. $\vec{w} = (w^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$) is an usc (resp. lsc) subsolution (resp. supersolution) of the system (2.1) such that, for any $(i,j) \in \Gamma^1 \times \Gamma^2$, both u^{ij} and w^{ij} belong to Π_g, i.e., there exist two constants γ and C such that*

$$\forall (i,j) \in \Gamma^1 \times \Gamma^2,\ \forall (t,x) \in [0,T] \times \mathbb{R}^k,\ |u^{ij}(t,x)| + |w^{ij}(t,x)| \leq C(1 + |x|^\gamma). \tag{4.5}$$

Then, it holds that for any $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$u^{ij}(t,x) \leq w^{ij}(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^k.$$

Proof. Let us proceed by contradiction and let $(u^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ (resp. $\vec{w}=(w^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$) be *usc* (resp. *lsc*) and a subsolution (resp. a supersolution) of the system (2.1) such that there exists $\varepsilon_0>0$ and $(t_0,x_0)\in[0,T]\times\mathbb{R}^k$ such that

$$\max_{i,j}\left((u^{ij}-w^{ij})(t_0,x_0)\right)\geq\epsilon_0. \tag{4.6}$$

Next, w.l.o.g. we may assume that for any $(i,j)\in\Gamma^1\times\Gamma^2$,

$$\lim_{|x|\to\infty}(u^{ij}-w^{ij})(t,x)=-\infty. \tag{4.7}$$

Indeed, if this is not the case, one may replace w^{ij} with $w^{ij,\vartheta,\mu}$ defined by

$$w^{ij,\vartheta,\mu}(t,x)=w^{ij}(t,x)+\vartheta e^{-\mu t}|x|^{2\gamma+2},(t,x)\in[0,T]\times\mathbb{R}^k,$$

which is still an *usc* supersolution of (2.1) for $\vartheta>0$ and $\mu\geq\mu_0$ which satisfies (4.7) (a proof of the supersolution property for good choices of ϑ and μ can be found in e.g. Pham (2009) (([25]), pp.76). Therefore, it suffices to show that $u^{ij}(t,x)\leq w^{ij,\vartheta,\mu}(t,x),\ (t,x)\in[0,T]\times\mathbb{R}^k$, since, by taking the limit as $\vartheta\to 0$, we deduce that $u^{ij}(t,x)\leq w^{ij}(t,x),\ (t,x)\in[0,T]\times\mathbb{R}^k$. Thus, assume that (4.6) and (4.7) are satisfied. Then, there exists $R>0$ such that

$$\begin{aligned}\max_{(t,x)\in[0,T]\times\mathbb{R}^k}\max_{i,j}\{(u^{ij}-w^{ij})(t,x)\}&=\max_{(t,x)\in[0,T]\times B(0,R)}\max_{i,j}\{(u^{ij}-w^{ij})(t,x)\}\\&=\max_{ij}(u^{ij}-w^{ij})(t^*,x^*)\geq\epsilon_0>0,\end{aligned} \tag{4.8}$$

where, $(t^*,x^*)\in[0,T)\times B(0,R)$, where, $B(0,R)$ denotes the ball in $\mathbb{R}^k$ with center the origin and radius R, since by definition $u^{ij}(T,x)\leq w^{ij}(T,x)$, for all $(i,j)\in\Gamma^1\times\Gamma^2$.

The remaining of the proof is obtained in two steps: the first step which is the main one establishes the comparison result under the additional condition (4.9) and the second step provides the proof in the general case.

<u>**Step 1.**</u> Let us make the following assumption on the functions $(f^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$. For all

$$\lambda>c(f^{i,j})(\Lambda-1),\ (i,j)\in\Gamma^1\times\Gamma^2,\ (t,x,\vec{y},z)\in[0,T]\times\mathbb{R}^k\times\mathbb{R}^{\Lambda+d},\ \text{ and }\ (u,v)\in\mathbb{R}^2\text{ s.t. }u\geq v,$$

$$f^{ij}(t,x,[\vec{y}^{-(ij)},u],z)-f^{ij}(t,x,[\vec{y}^{-(ij)},v],z)\leq-\lambda(u-v), \tag{4.9}$$

where, $c(f^{ij})$ is the Lipschitz constant of f^{ij} w.r.t. $(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$. Next, let (i_0,j_0) be an element of $\tilde{\Gamma}(t^*,x^*)$ that satisfies (4.1). For $n\geq 1$, let $\Phi_n^{i_0,j_0}$ be the function defined as follows.

$$\Phi_n^{i_0,j_0}(t,x,y):=(u^{i_0j_0}(t,x)-w^{i_0j_0}(t,y))-\phi_n(t,x,y),\quad (t,x,y)\in[0,T]\times\mathbb{R}^{k+k},$$

where,

$$\phi_n(t,x,y):=n|x-y|^{2\gamma+2}+|x-x^*|^{2\gamma+2}+(t-t^*)^2.$$

Since $\Phi_n^{i_0,j_0}$ is *usc* in (t,x,y), there exists $(t_n,x_n,y_n)\in[0,T]\times B(0,R)^2$ such that

$$\Phi_n^{i_0,j_0}(t_n,x_n,y_n)=\max_{(t,x,y)\in[0,T]\times B(0,R)^2}\Phi_n^{i_0,j_0}(t,x,y).$$

Moreover,

$$\begin{aligned}\Phi_n^{i_0,j_0}(t^*,x^*,x^*)=u^{i_0,j_0}(t^*,x^*)-w^{i_0,j_0}(t^*,x^*)&\leq u^{i_0,j_0}(t^*,x^*)-w^{i_0,j_0}(t^*,x^*)+\phi_n(t_n,x_n,y_n)\\&\leq u^{i_0,j_0}(t_n,x_n)-w^{i_0,j_0}(t_n,y_n).\end{aligned} \tag{4.10}$$

The definition of ϕ_n together with the growth condition of u^{ij} and w^{ij} implies that $(x_n - y_n)_{n\geq 1}$ converges to 0. Next, for any subsequence $((t_{n_l}, x_{n_l}, y_{n_l}))_l$ which converges to $(\tilde{t}, \tilde{x}, \tilde{x})$ we deduce from (4.10) that

$$u^{i_0 j_0}(t^*, x^*) - w^{i_0 j_0}(t^*, x^*) \leq u^{i_0, j_0}(\tilde{t}, \tilde{x}) - w^{i_0 j_0}(\tilde{t}, \tilde{x}),$$

since $u^{i_0 j_0}$ is *usc* and $w^{i_0 j_0}$ is *lsc*. As the maximum of $u^{i_0 j_0} - w^{i_0 j_0}$ on $[0,T] \times B(0,R)$ is reached in (t^*, x^*), then this last inequality is actually an equality. Using the definition of ϕ_n and (4.10), we deduce that the sequence $((t_n, x_n, y_n))_n$ converges to (t^*, x^*, x^*) which also implies

$$n|x_n - y_n|^{2\gamma+2} \to 0 \quad \text{and} \quad (u^{i_0 j_0}(t_n, x_n), w^{i_0 j_0}(t_n, y_n)) \to (u^{i_0 j_0}(t^*, x^*), w^{i_0 j_0}(t^*, x^*)),$$

as $n \to \infty$. This latter convergence holds, since we first obtain from (4.10) and in taking into account that $u^{i_0 j_0}$ and $w^{i_0 j_0}$ are *lsc* and *usc* respectively,

$$\begin{aligned} u^{i_0 j_0}(t^*, x^*) - w^{i_0 j_0}(t^*, x^*) \quad &\leq \underline{\lim}_n (u^{i_0 j_0}(t_n, x_n) - w^{i_0 j_0}(t_n, y_n)) \leq \overline{\lim}_n (u^{i_0 j_0}(t_n, x_n) - w^{i_0 j_0}(t_n, y_n)) \\ &\leq \overline{\lim}_n u^{i_0 j_0}(t_n, x_n) - \underline{\lim}_n w^{i_0 j_0}(t_n, y_n)) \leq u^{i_0 j_0}(t^*, x^*) - w^{i_0 j_0}(t^*, x^*). \end{aligned}$$

Thus the sequence $(u^{i_0 j_0}(t_n, x_n) - w^{i_0 j_0}(t_n, y_n))_{n\geq 0}$ converges to $u^{i_0 j_0}(t^*, x^*) - w^{i_0 j_0}(t^*, x^*)$ as $n \to \infty$. Then

$$\underline{\lim}_n u^{i_0 j_0}(t_n, x_n) = \quad u^{i_0 j_0}(t^*, x^*) - w^{i_0 j_0}(t^*, x^*) + \underline{\lim}_n w^{i_0 j_0}(t_n, y_n) \geq u^{i_0 j_0}(t^*, x^*) \geq \overline{\lim}_n u^{i_0 j_0} t_n, x_n).$$

It follows that the sequence $(u^{i_0 j_0}(t_n, x_n))_n$ converges to $u^{i_0 j_0}(t^*, x^*)$ and then $(w^{i_0 j_0}(t_n, y_n))_n$ converges also to $w^{i_0 j_0}(t^*, x^*)$.

Next, recalling that $u^{i_0 j_0}$ (resp. $w^{i_0 j_0}$) is *usc* (resp. *lsc*) and satisfies (4.1), then, for n large enough and at least for a subsequence which we still index by n, we obtain

$$u^{i_0 j_0}(t_n, x_n) > \max_{k \in (\Gamma^1)^{-i}} (u^{k j_0}(t_n, x_n) - g_{i_0 k}(t_n, x_n)), \tag{4.11}$$

and

$$w^{i_0 j_0}(t_n, x_n) < \min_{l \in (\Gamma^2)^{-j}} (w^{i_0 l}(t_n, x_n) - g_{j_0 l}(t_n, x_n)). \tag{4.12}$$

Applying now Crandall-Ishii-Lions's Lemma (see e.g. [5] or [10], pp.216) with $\Phi_n^{i_0 j_0}$ and ϕ_n at the point (t_n, x_n, y_n) (for n large enough in such a way that this latter triple will belong to $[0,T] \times B(0,R)^2$), there exist $(p_u^n, q_u^n, M_u^n) \in \bar{J}^{2,+}(u^{i_0 j_0})(t_n, x_n)$ and $(p_w^n, q_w^n, M_w^n) \in \bar{J}^{2,-}(w^{i_0 j_0})(t_n, y_n)$ such that

$p_u^n - p_w^n = \partial_t \tilde{\varphi}_n(t_n, x_n, y_n) = 2(t_n - t^*)$, q_u^n (resp. q_w^n) $= \partial_x \varphi_n(t_n, x_n, y_n)$ (resp. $-\partial_y \varphi_n(t_n, x_n, y_n)$) and

$$\begin{pmatrix} M_u^n & 0 \\ 0 & -N_w^n \end{pmatrix} \leq A_n + \frac{1}{2n} A_n^2, \tag{4.13}$$

where, $A_n = D^2_{(x,y)} \varphi_n(t_n, x_n, y_n)$. Taking into account that $(u^{ij})_{i,j}$ (resp. $(w^{ij})_{i,j}$) is a subsolution (resp. supersolution) of (2.1) and using once more (4.11) and (4.12) we get

$$-p_u^n - b(t_n, x_n)^\top . q_u^n - \frac{1}{2} Tr[(\sigma\sigma^\top)(t_n, x_n) M_u^n] - f^{i_0 j_0}(t_n, x_n, (u^{ij}(t_n, x_n))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma(t_n, x_n)^\top . q_u^n) \leq 0,$$

and

$$-p_w^n - b(t_n, y_n)^\top . q_w^n - \frac{1}{2} Tr[(\sigma\sigma^\top)(t_n, y_n) M_w^n] - f^{i_0 j_0}(t_n, y_n, (w^{ij}(t_n, y_n))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma(t_n, y_n)^\top . q_w^n) \geq 0.$$

Taking the difference between these two inequalities yields

$$\begin{aligned} -(p_u^n - p_w^n) - (b(t_n, x_n)^\top . q_u^n - b(t_n, y_n)^\top . q_w^n) - \tfrac{1}{2} Tr[\{\sigma\sigma^\top(t_n, x_n) M_u^n - \sigma\sigma^\top(t_n, y_n) M_w^n\}] \\ -\{f^{i_0 j_0}(t_n, x_n, (u^{ij}(t_n, x_n))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma(t_n, x_n)^\top \cdot q_u^n) \\ -f^{i_0 j_0}(t_n, y_n, (w^{ij}(t_n, y_n))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma(t_n, y_n)^\top . q_w^n)\} \leq 0, \end{aligned} \tag{4.14}$$

and then

$$-\{f^{i_0j_0}(t_n,x_n,(u^{ij}(t_n,x_n))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma(t_n,x_n)^\top.q_u^n)$$
$$-f^{i_0j_0}(t_n,y_n,(w^{ij}(t_n,y_n))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma(t_n,y_n)^\top.q_w^n)\}\le\varrho_n,$$

with $\overline{\lim}_{n\to\infty}\varrho_n\le 0$, using the fact that all the terms in the first line of (4.14) are converging sequences. Linearizing $f^{i_0j_0}$ (see Appendix A3), which is Lipschitz w.r.t. $(y^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$, and using Assumption (4.9), we obtain

$$\lambda(u^{i_0j_0}(t_n,x_n)-w^{i_0j_0}(t_n,y_n))-\sum_{(i,j)\in\Gamma^1\times\Gamma^2,(i,j)\neq(i_0j_0)}\Theta_n^{i,j}(u^{ij}(t_n,x_n)-w^{ij}(t_n,y_n))\le\varrho_n,$$

where, $\Theta_n^{i,j}$ is the increment rate of $f^{i_0j_0}$ w.r.t. y^{ij} which is uniformly bounded (w.r.t. n) and non-negative thanks to the monotonicity assumption (H2). Therefore,

$$\begin{aligned}\lambda(u^{i_0j_0}(t_n,x_n) &-w^{i_0j_0}(t_n,y_n))\\ &\le\textstyle\sum_{(i,j)\in\Gamma^1\times\Gamma^2,(i,j)\neq(i_0,j_0)}\Theta_n^{i,j}(u^{ij}(t_n,x_n)-w^{ij}(t_n,y_n))+\varrho_n\\ &\le c(f^{i_0j_0})\times\textstyle\sum_{(i,j)\in\Gamma^1\times\Gamma^2,(i,j)\neq(i_0,j_0)}((u^{ij}(t_n,x_n)-w^{ij}(t_n,y_n))^++\varrho_n.\end{aligned}$$

Taking the limit as $n\to\infty$ we obtain

$$\begin{aligned}\lambda(u^{i_0j_0}(t^*,x^*) &-w^{i_0j_0}(t^*,y^*))\\ &\le\overline{\lim}_{n\to\infty}c(f^{i_0j_0})\left(\textstyle\sum_{(i,j)\in\Gamma^1\times\Gamma^2,(i,j)\neq(i_0,j_0)}(u^{ij}(t_n,x_n)-w^{ij}(t_n,y_n))^+\right)\\ &\le c(f^{i_0j_0})\left(\textstyle\sum_{(i,j)\in\Gamma^1\times\Gamma^2,(i,j)\neq(i_0,j_0)}\overline{\lim}_{n\to\infty}(u^{ij}(t_n,x_n)-w^{ij}(t_n,y_n))^+\right)\\ &\le c(f^{i_0j_0})\left(\textstyle\sum_{(i,j)\in\Gamma^1\times\Gamma^2,(i,j)\neq(i_0,j_0)}(u^{ij}(t^*,x^*)-w^{ij}(t^*,x^*))^+\right)\end{aligned},$$

since u^{ij} (resp. w^{ij}) is *usc* (resp. *lsc*). As (i_0,j_0) belongs to $\tilde\Gamma(t^*,x^*)$, we obtain

$$\lambda(u^{i_0j_0}(t^*,x^*)-w^{i_0j_0}(t^*,y^*))\le c(f^{i_0j_0})\left((\Lambda-1)(u^{i_0j_0}(t^*,x^*)-w^{i_0j_0}(t^*,x^*))\right).$$

But this is contradictory with (4.8) and (4.9). Thus, for any $(i,j)\in\Gamma^1\times\Gamma^2$, $u^{ij}\le w^{ij}$. □

Step 2. The general case.
For arbitrary $\lambda\in\mathbb{R}$, if $(u^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2\in\Gamma^1\times\Gamma^2}$ (resp. $(w^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$) is a subsolution (resp. supersolution) of (2.1) then $\tilde u^{ij}(t,x)=e^{\lambda t}u^{ij}(t,x)$ and $\tilde w^{ij}(t,x)=e^{\lambda t}w^{ij}(t,x)$ is a subsolution (resp. supersolution) of the following system of variational inequalities with oblique reflection. For every $(i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases}\min\{\tilde v^{ij}(t,x)-\max_{l\in(\Gamma^1)^{-i}}\{\tilde v^{lj}(t,x)-e^{\lambda t}\underline{g}_{il}(t,x)\};\\ \qquad\max\{\tilde v^{ij}(t,x)-\min_{k\in(\Gamma^2)^{-j}}\{e^{\lambda t}\bar g_{jk}(t,x)+\tilde v^{ij}(t,x)\};-\partial_t\tilde v^{ij}(t,x)-\mathcal{L}\tilde v^{ij}(t,x)+\lambda\tilde v^{ij}(t,x)\\ \qquad\qquad-e^{\lambda t}f^{ij}(t,x,(e^{-\lambda t}\tilde v^{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},e^{-\lambda t}\sigma^\top(t,x).D_x\tilde v^{ij}(t,x))\}=0,\\ \tilde v^{ij}(T,x)=e^{\lambda T}h^{ij}(x).\end{cases}\tag{4.15}$$

But, by choosing λ large enough the functions

$$F^{ij}(t,x,(u^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z)=-\lambda u^{ij}+e^{\lambda t}f^{ij}(t,x,(e^{-\lambda t}u^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},e^{-\lambda t}z),\ \ (i,j)\in\Gamma^1\times\Gamma^2,$$

satisfy Condition (4.9). Hence, thanks to the result stated in Step 1, we have $\tilde u^{ij}\le\tilde v^{ij}$, $(i,j)\in\Gamma^1\times\Gamma^2$. Thus, $u^{ij}\le v^{ij}$, for any $(i,j)\in\Gamma^1\times\Gamma^2$, which is the desired result. ■

As a consequence of this comparison result, we obtain the following one related to uniqueness of the solution of (2.1).

Corollary 2. *If the system (2.1) admits a viscosity solution in the class Π_g, then it is unique and continuous.*

Proof. Indeed, assume that $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a solution of (2.1) that belongs to Π_g. Then, thanks to the previous comparison result, for any $(i,j)\in\Gamma^1\times\Gamma^2$ we have $v^{ij,*}\le v^{ij}_*$. Thus, $v^{ij,*}=v^{ij}_*$ and then $v^{ij}=v^{ij}_*=v^{ij,*}$, which means that v^{ij} is continuous. Next, if $(\hat{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is another solution of (2.1) in the class Π_g, then it is also continuous and by the Comparison Theorem 4.2 we have $v^{ij}\le\hat{v}^{ij}$ and $v^{ij}\ge\hat{v}^{ij}$. Hence, $v^{ij}=\hat{v}^{ij}$, $(i,j)\in\Gamma^1\times\Gamma^2$, i.e., uniqueness of the solution of (2.1). ■

5 Viscosity solution of the system (2.1)

In this section we prove that the family $(\bar{v}^{ij})_{i,j}$ constructed in Section 3 provides the unique continuous solution in viscosity sense of the system (2.1). For sake of clarity, the proof is divided into several steps.

Proposition 5.1. *The family $(\bar{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a viscosity subsolution of the system (2.1).*

Proof. First recall that for each $(i,j)\in\Gamma^1\times\Gamma^2$, $\bar{v}^{ij}$ is *usc*, since $\bar{v}^{ij}=\lim_m\searrow v^{ij,m}$, where $v^{ij,m}$ is a continuous function solution of the system (3.8). Thus, for any $(i,j)\in\Gamma^1\times\Gamma^2$, it holds that $\bar{v}^{ij,*}=\bar{v}^{ij}$. Next, at T we have

$$\bar{v}^{ij}(T,x)=\lim_m\searrow v^{ij,m}(T,x)=h^{ij}(x),\quad x\in\mathbb{R}^k.$$

We shall now prove that, for any (t,x) in $[0,T)\times\mathbb{R}^k$, any $((i,j)\in\Gamma^1\times\Gamma^2$ and $(\underline{p},\underline{q},\underline{M})$ in $\bar{J}^+\bar{v}^{ij}(t,x)$,

$$\begin{aligned}&\min[\big(\bar{v}^{ij}-\bar{L}^{ij}\big)(t,x),\max\{\big(\bar{v}^{ij}-\bar{U}^{ij}\big)(t,x),\\&\quad-\underline{p}-b(t,x)\underline{q}-\tfrac12\mathrm{Tr}(\sigma\sigma^T(t,x)\underline{M})-f^{ij}(t,x,(\bar{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x).\underline{q})\}]\le 0,\end{aligned}\tag{5.1}$$

with $\bar{L}^{ij}$ and $\bar{U}^{ij}$ defined as follows:

$$\bar{L}^{ij}(t,x)=\max_{k\in(\Gamma^1)^{-i}}\big(\bar{v}^{kj}(t,x)-\underline{g}_{ik}(t,x)\big)\ \text{ and }\ \bar{U}^{ij}(t,x)=\min_{l\in(\Gamma^2)^{-j}}\big(\bar{v}^{il}(t,x)+\bar{g}_{jl}(t,x)\big).$$

Now, let $(i,j)\in\Gamma^1\times\Gamma^2$ be fixed. Then it is equivalent to show that, either

$$\big(\bar{v}^{ij}-\bar{L}^{ij}\big)(t,x)\le 0,\tag{5.2}$$

or

$$\max\{\big(\bar{v}^{ij}-\bar{U}^{ij}\big)(t,x),-\underline{p}-b(t,x)\underline{q}-\frac12\mathrm{Tr}(\sigma\sigma^T(t,x)\underline{M})-f^{ij}(t,x,\vec{v}(t,x),\sigma^\top(t,x).\underline{q})\}\le 0.\tag{5.3}$$

If (5.2) is satisfied then the subsolution property (5.1) holds. Therefore, from now on, we suppose that there exists $\epsilon_0>0$ such that

$$\bar{v}^{ij}(t,x)\ge\bar{L}^{ij}(t,x)+\epsilon_0,\tag{5.4}$$

and show (5.3). Thanks to the decreasing convergence of $(v^{ij,m})_{m\ge0}$ to $\bar{v}^{ij}$, $(i,j)\in\Gamma^1\times\Gamma^2$, there exists m_0 such that for any $m\ge m_0$, we have

$$v^{ij,m}(t,x)\ge L^{ij,m}(t,x)+\frac{\epsilon_0}{2}.\tag{5.5}$$

Next, by continuity of $v^{ij,m}$ and $L^{ij,m}$, there exists a neighborhood $\mathcal{O}_m$ of (t,x) such that

$$v^{ij,m}(t',x')\ge L^{ij,m}(t',x')+\frac{\epsilon_0}{4},\quad (t',x')\in\mathcal{O}_m.\tag{5.6}$$

Now, by Lemma 6.1 in [5] there exists a subsequence $((t_k,x_k))_{k\ge0}$ such that

$$(t_k,x_k)\to_{k\to\infty}(t,x)\ \text{ and }\ \bar{v}^{ij}(t,x)=\lim_{k\to\infty}v^{ij,k}(t_k,x_k).\tag{5.7}$$

Moreover, there exists a sequence $(p_k, q_k, M_k) \in \bar{J}^+\big(v^{ij,k}(t_k, x_k)\big)$ such that

$$\lim_{k\to\infty} (p_k, q_k, M_k) = (\underline{p}, \underline{q}, \underline{M}). \tag{5.8}$$

But, the subsequence $((t_k, x_k))_{k\geq 0}$ can be chosen in such a way that for any $k \geq 0$, $(t_k, x_k) \in \mathcal{O}_k$. Applying now the viscosity subsolution property of $v^{ij,k}$ (which satisfies (3.8)) at (t_k, x_k) and taking into account of (5.6) we obtain

$$-p_k - b(t_k, x_k)^\top.q_k - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t_k, x_k)M_k) - \bar{f}^{ij,k}(t_k, x_k, (\bar{v}^{pq,k}(t_k, x_k))_{(p,q)\in\Gamma^1\times\Gamma^2}, \sigma(t_k, x_k)^\top q_k) \leq 0, \tag{5.9}$$

where, once more,

$$\bar{f}^{ij,k}(s, x, (y^{pq})_{(p,q)\in\Gamma^1\times\Gamma^2}, z) = f^{ij}(s, x, (y^{pq})_{(p,q)\in\Gamma^1\times\Gamma^2}, z) - k\big(y^{ij} - \min_{l\in(\Gamma^2)^{-j}}(y^{il} + \bar{g}_{jl}(s, x))\big)^+.$$

Next, thanks to the boundedness of the sequence $((t_k, x_k))_{k\geq 0}$, the uniform polynomial growth of $\bar{v}^{pq,k}$ $k \geq 0$, (by Proposition 3.2 and Corollary 1), the assumptions (H0)-(H2) on b, σ and f^{ij}, and the convergence of $((p_k, q_k, M_k))_k$ to $(\underline{p}, \underline{q}, \underline{M})$, we deduce from (5.9) that

$$\epsilon_k := \big(\bar{v}^{ij,k}(t_k, x_k) - \min_{l\in(\Gamma^2)^{-j}} \big(\bar{v}^{il,k}(t_k, x_k) + \bar{g}_{jl}(t_k, x_k)\big)\big)^+ \to 0, \quad k \to \infty.$$

But, for any fixed (t, x) and k_0, the sequence $(\bar{v}^{il,k}(t, x))_{k\geq k_0}$ is decreasing and then for any $k \geq k_0 \geq m_0$,

$$\bar{v}^{ij,k}(t_k, x_k) \leq \min_{l\neq j} \big(\bar{v}^{il,k}(t_k, x_k) + \bar{g}_{jl}(t_k, x_k)\big) + \epsilon_k \leq \min_{l\neq j} \big(\bar{v}^{il,k_0}(t_k, x_k) + \bar{g}_{jl}(t_k, x_k)\big) + \epsilon_k.$$

Taking now the limit as $k \to +\infty$, in view of the continuity of $\bar{v}^{il,k_0}$, we get

$$\lim_k \bar{v}^{ij,k}(t_k, x_k) = \bar{v}^{ij}(t, x) \leq \min_{j\neq l}(v^{il,k_0}(t, x) + \bar{g}_{jl}(t, x)).$$

Finally, passing to the limit as k_0 goes to $+\infty$ to obtain

$$\bar{v}^{ij}(t, x) \leq \min_{j\neq l}(\bar{v}^{il}(t, x) + \bar{g}_{jl}(t, x)) = \bar{U}^{ij}(t, x). \tag{5.10}$$

Let us now consider a subsequence of (k), which we denote by (k_l), such that for any $(p, q) \in \Gamma^1 \times \Gamma^2$, the sequence $(\bar{v}^{pq,k}(t_{k_l}, x_{k_l}))_l$ is convergent. This subsequence exists since the functions $\bar{v}^{ij,k_l}$ are uniformly of polynomial growth (by Proposition 3.2 and Corollary 1). Then, taking the limit w.r.t. l in equation (5.9), we obtain

$$\begin{aligned} -\underline{p} - \underline{q}b(t, x) - \tfrac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t, x)\underline{M}) \quad &\leq \overline{\lim}_{l\to\infty} \bar{f}^{ij,k_l}(t_{k_l}, x_{k_l}, (\bar{v}^{pq,k_l}(t_{k_l}, x_{k_l}))_{(p,q)\in\Gamma^1\times\Gamma^2}, \sigma(t_{k_l}, x_{k_l})^\top q_{k_l}) \\ &\leq \overline{\lim}_{l\to\infty} f^{ij}(t_{k_l}, x_{k_l}, (\bar{v}^{pq,k_l}(t_{k_l}, x_{k_l}))_{(p,q)\in\Gamma^1\times\Gamma^2}, \sigma(t_{k_l}, x_{k_l})^\top q_{k_l}) \\ &= f^{ij}(t, x, (\lim_{l\to\infty} \bar{v}^{pq,k_l}(t_{k_l}, x_{k_l}))_{(p,q)\in\Gamma^1\times\Gamma^2}, \sigma(t, x)^\top \underline{q}), \end{aligned} \tag{5.11}$$

since f^{ij} is continuous in $(t, x, \vec{y}, z)$. Now for any $(p, q) \in \Gamma^1 \times \Gamma^2$, since $\bar{v}^{pq,n}$ is continuous and $\bar{v}^{pq,n} \geq \bar{v}^{pq,n+1}$, $\forall n \geq 0$, it holds that

$$\bar{v}^{pq,*}(t, x) = \bar{v}^{pq}(t, x) = \overline{\lim_{t'\to t, x'\to x, n\to\infty}} \bar{v}^{pq,n}(t', x'), \qquad (t, x) \in [0, T] \times \mathbb{R}^k.$$

Therefore, for any $(p, q) \in \Gamma^1 \times \Gamma^2$ $((p, q) \neq (i, j))$, we have

$$\bar{v}^{pq}(t, x) \geq \lim_{l\to\infty} \bar{v}^{pq,k_l}(t_{k_l}, x_{k_l}) \text{ and } \bar{v}^{ij}(t, x) = \lim_{l\to\infty} \bar{v}^{ij,k_l}(t_{k_l}, x_{k_l}). \tag{5.12}$$

As f^{ij} is non-decreasing w.r.t. y^{kl}, $(k, l) \in \Gamma^1 \times \Gamma^2, (k, l) \neq (i, j)$, we deduce from (5.11) and (5.12) that

$$-\underline{p} - \underline{q}b(t, x) - \tfrac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t, x)\underline{M}) \leq f^{ij}(t, x, (\bar{v}^{pq}(t, x))_{(p,q)\in\Gamma^1\times\Gamma^2}, \sigma(t, x)^\top \underline{q}). \tag{5.13}$$

Finally, under the condition (5.4), the relations (5.13), (5.10) imply that (5.3) is satisfied. Thus $\bar{v}^{ij}$ is a viscosity subsolution for

$$\begin{cases} \min[\big(\bar{v}^{ij} - \underline{L}^{ij}\big)(t,x), \max\{\big(\bar{v}^{ij} - \underline{U}^{ij}\big)(t,x), \\ \qquad -\underline{p} - b(t,x)^\top \underline{q} - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)\underline{M}) - f^{ij}(t,x,(\bar{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(t,x).\underline{q})\}] = 0, \\ \bar{v}^{ij}(T,x) = h^{ij}(x). \end{cases}$$

Since (i,j) is arbitrary, $(\bar{v}^{ij})_{i,j\in\Gamma^1\times\Gamma^2}$ is a viscosity subsolution for (2.1). This finishes the proof. ∎

Proposition 5.2. *Let m_0 be fixed in $\mathbb{N}$. Then, the family $(\bar{v}^{ij,m_0})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a viscosity supersolution of the system (2.1).*

Proof. First and thanks to Proposition 3.2, we know that the triples $((\bar{Y}^{ij,m_0}, \bar{Z}^{ij,m_0}, \bar{K}^{ij,m_0}))_{(i,j)\in\Gamma^1\times\Gamma^2}$ introduced in (3.5), solve the following system of reflected BSDEs: For every $(i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases} \bar{Y}^{ij,m_0} \in \mathcal{S}^{2,1},\ \bar{Z}^{ij,m_0}\in\mathcal{H}^{2,d} \text{ and } \bar{K}^{ij,m_0}\in\mathcal{A}^{2,+}\ ; \\ \bar{Y}^{ij,m_0}_s = h^{ij}(X^{t,x}_T) + \int_s^T \bar{f}^{ij,m_0}(r, X^{t,x}_r, (\bar{Y}^{kl,m_0}_r)_{(k,l)\in\Gamma^1\times\Gamma^2}, \bar{Z}^{ij,m_0}_r)dr + \bar{K}^{ij,m_0}_T - \bar{K}^{ij,m_0}_s - \int_s^T \bar{Z}^{ij,m_0}_r dB_r \\ \bar{Y}^{ij,m_0}_s \geq \max\limits_{k\in(\Gamma^1)^{-i}}\{\bar{Y}^{kj,m_0}_s - \underline{g}_{ik}(s,X^{t,x}_s)\},\ s\leq T\,; \\ \int_0^T(\bar{Y}^{ij,m_0}_s - \max_{k\in(\Gamma^1)^{-i}}\{\bar{Y}^{kj,m_0}_s - \underline{g}_{ik}(s,X^{t,x}_s)\})d\bar{K}^{ij,m_0}_s = 0 \end{cases} \tag{5.14}$$

where, for any $(i,j)\in\Gamma^1\times\Gamma^2$ and $(s,\vec{y},z^{ij})$,

$$\bar{f}^{ij,m_0}(s,X^{t,x}_s,\vec{y},z^{ij}) = f^{i,j}(s,X^{t,x}_s,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z^{ij}) - m_0\big(y^{ij} - \min_{l\in(\Gamma^2)^{-j}}(y^{il} + \bar{g}_{jl}(s,X^{t,x}_s))\big)^+.$$

Furthermore, it holds true that

$$\forall (i,j)\in\Gamma^1\times\Gamma^2,\ \forall (t,x)\in[0,T]\times\mathbb{R}^k,\ \forall s\in[t,T],\ \bar{Y}^{ij,m_0}_s = \bar{v}^{ij,m_0}(s,X^{t,x}_s).$$

On the other hand, we note that $\bar{Y}^{ij,m_0}$ is the value function of a zero-sum Dynkin game (see Appendix A4), i.e., it satisfies, for all $s\leq T$

$$\begin{aligned} \bar{Y}^{ij;m_0}_s = &\operatorname{ess\,sup}_{\sigma\geq s}\operatorname{ess\,inf}_{\tau\geq s}\mathbb{E}[\textstyle\int_s^{\sigma\wedge\tau} f^{ij}(r,X^{t,x}_r,(\bar{Y}^{ij;m_0}_r)_{(i,j)\in\Gamma^1\times\Gamma^2},\bar{Z}^{ij;m_0}_r)dr + \\ &\max_{k\in(\Gamma^1)^{-i}}\{\bar{Y}^{kj;m_0}_\sigma - \underline{g}_{ik}(\sigma,X^{t,x}_\sigma)\}\mathbb{1}_{[\sigma<\tau]} + \{\bar{Y}^{il;m_0}_\tau \wedge \min_{l\in(\Gamma^2)^{-j}}\{\bar{Y}^{il;m_0}_\tau - \bar{g}_{jl}(\tau,X^{t,x}_\tau)\}\}\mathbb{1}_{[\tau\leq\sigma<T]} \\ &+h^{ij}(X^{t,x}_T)\mathbb{1}_{[\tau=\sigma=T]}|\mathcal{F}_s]. \end{aligned} \tag{5.15}$$

Thus, by Theorem 3.7 in Hamadène-Hassani (2005) ([15]), it follows that $\bar{v}^{ij,m_0}$ is the unique viscosity solution for the following PDE with two obstacles.

$$\begin{cases} \min[\vartheta(t,x) - \max_{k\in(\Gamma^1)^{-i}}\{\bar{v}^{kj,m_0}(t,x) - \underline{g}_{ik}(t,x)\}, \max\{\vartheta(t,x) - \bar{v}^{ij,m_0}(t,x)\vee\min_{l\in(\Gamma^2)^{-j}}(\bar{v}^{il,m_0}(t,x) - \bar{g}_{jl}(t,x)), \\ \qquad -\partial_t\vartheta(t,x) - b(t,x)^\top D_x\vartheta(t,x) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)D^2_{xx}\vartheta(t,x)) - \\ \qquad\qquad f^{ij}(t,x,(\bar{v}^{kl,m_0}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x).D_x\vartheta(t,x))\}] = 0\,; \\ \vartheta(T,x) = h^{ij}(x). \end{cases}$$

Next, let $(t,x)\in[0,T]\times\mathbb{R}^k$ and $(p,q,M)\in\bar{J}^-[\bar{v}^{ij,m_0}](t,x)$. As $\bar{v}^{ij,m_0}$ is a solution in a viscosity sense of the previous PDE with two obstacles then it holds that

$$\bar{v}^{ij,m_0}(t,x) \geq \max_{k\in(\Gamma^1)^{-i}}\{\bar{v}^{kj,m_0}(t,x) - \underline{g}_{ik}(t,x)\} \tag{5.16}$$

and

$$\begin{aligned} &\max\{\bar{v}^{ij,m_0}(t,x) - \bar{v}^{ij,m_0}(t,x)\vee\min_{l\in(\Gamma^2)^{-j}}(\bar{v}^{il,m_0}(t,x) - \bar{g}_{jl}(t,x))\,; \\ &\qquad -p - b(t,x)^\top q - \tfrac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)M) - f^{ij}(t,x,(\bar{v}^{kl,m_0}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x).q)\} \geq 0. \end{aligned} \tag{5.17}$$

But, for any constants $a, b \in \mathbb{R}$ we have $a - (a \vee b) \leq a - b$ and thus $a - (a \vee b) \geq 0 \Rightarrow a - b \geq 0$. Therefore, we have from (5.17),

$$\begin{aligned}&\max\{\bar{v}^{ij,m_0}(t,x) - \min_{l\in(\Gamma^2)^{-j}}(\bar{v}^{il,m_0}(t,x) - \bar{g}_{jl}(t,x))\,;\\ &\qquad -p - b(t,x)^\top q - \tfrac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)M) - f^{ij}(t,x,(\bar{v}^{kl,m_0}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(t,x).q)\} \geq 0.\end{aligned}$$

Combining this inequality with (5.16) and since $v^{ij,m_0}(T,x) = h^{ij}(x)$ it follows that v^{ij,m_0} is a viscosity supersolution of the system

$$\begin{cases} \min[\vartheta(t,x) - \max_{k\in(\Gamma^1)^{-i}}(\bar{v}^{kj,m_0}(t,x) - \underline{g}_{ik}(t,x))\,;\, \max\{\vartheta(t,x) - \min_{l\in(\Gamma^2)^{-j}}(\bar{v}^{il,m_0}(t,x) - \bar{g}_{jl}(t,x));\\ \qquad -\partial_t\vartheta(t,x) - b(t,x)^\top D_x\vartheta(t,x) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)D^2_{xx}\vartheta(t,x)) - \\ \qquad\qquad f^{ij}(t,x,(\bar{v}^{kl,m_0}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(t,x).D_x\vartheta(t,x))\}] = 0,\\ \vartheta(T,x) = h^{ij}(x). \end{cases}$$

Since (i,j) is arbitrary in $\Gamma^1 \times \Gamma^2$, the system of continuous functions $(v^{ij,m_0})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a supersolution of (2.1). ■

Consider now the set $\mathcal{U}_{m_0}$ defined as follows.

$$\mathcal{U} = \{\vec{u} := (u^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2} \text{ s.t. } \vec{u} \text{ is a subsolution of (2.1) and } \forall\,(i,j)\in\Gamma^1\times\Gamma^2,\ \bar{v}^{i,j} \leq u^{i,j} \leq \bar{v}^{ij,m_0}\}.$$

Obviously, $\mathcal{U}_{m_0}$ is not empty since it contains $(\bar{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$. Next for $(t,x)\in[0,T]\times\mathbb{R}^k$ and $(i,j)\in\Gamma^1\times\Gamma^2$, let us set:

$$^{m_0}v^{ij}(t,x) = \sup\{u^{ij}(t,x),\ (u^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2} \in \mathcal{U}_{m_0}\}.$$

We now state the main result of this section.

Theorem 5.3. *The family $(^{m_0}v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ does not depend on m_0 and is the unique continuous viscosity solution in the class Π_g of the system (2.1).*

Proof. We first note that for any $(i,j)\in\Gamma^1\times\Gamma^2$, $\bar{v}^{ij} \leq {}^{m_0}v^{i,j} \leq \bar{v}^{ij,m_0}$. Since $\bar{v}^{ij}$ and $\bar{v}^{ij,m_0}$ are of polynomial growth, then $(^{m_0}v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ belongs to Π_g.

The remaining of the proof is divided into two steps and mainly consists in adapting the Perron's method (see Crandall-Ishii-Lions, [5] Theorem 4.1, pp 23) to construct a viscosity solution to our general system of PDEs. To ease notation, we denote $(^{m_0}v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ by $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ as no confusion is possible.

Step 1. We first show that $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a subsolution of (2.1). Indeed, it is clear that for any $(t,x)\in[0,T]\times\mathbb{R}^k$, $\bar{v}^{ij}(t,x) \leq v^{ij}(t,x) \leq \bar{v}^{ij,m_0}(t,x)$. This implies that $\bar{v}^{ij} \leq v^{ij,*} \leq \bar{v}^{ij,m_0}$ since, as pointed out previously, $\bar{v}^{ij}$ is *usc* and v^{ij,m_0} is continuous. Therefore, for any $x\in\mathbb{R}^k$, we have $v^{ij,*}(T,x) = h^{ij}(x)$, since $\bar{v}^{ij}(T,x) = \bar{v}^{ij,m_0}(T,x) = h^{ij}(x)$.

Next, fix $(i,j)\in\Gamma^1\times\Gamma^2$ and let $(\tilde{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ be an arbitrary element of $\mathcal{U}_{m_0}$. Then, for any $(t,x)\in[0,T)\times\mathbb{R}^k$ and any $(p,q,M)\in\bar{J}^+\tilde{v}^{i,j,*}(t,x)$ we have

$$\begin{cases} \min[\big(\tilde{v}^{ij,*} - L^{ij}((\tilde{v}^{kl,*})_{k,l})\big)(t,x), \max\{\big(\tilde{v}^{ij,*} - U^{ij}((\tilde{v}^{kl,*})_{k,l})\big)(t,x)\,,\\ \qquad -p - b(t,x)^\top q - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)M) - f^{ij}(t,x,(\tilde{v}^{kl,*}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(t,x)q)\}] \leq 0. \end{cases}$$

By definition we have $\tilde{v}^{kl} \leq v^{kl}$ and then $\tilde{v}^{kl,*} \leq v^{kl,*}$ for any $(k,l)\in\Gamma^1\times\Gamma^2$. Since the operators $\vec{w} = (w^{kl})_{k,l} \mapsto \tilde{v}^{ij*} - L^{i,j}((w^{kl})_{k,l})$, $\vec{w} = (w^{kl})_{k,l} \mapsto \tilde{v}^{ij*} - U^{ij}((w^{kl})_{k,l})$ are decreasing, in view of the monotonicity property (H2) of the generator f^{ij}, we have

$$\begin{cases} \min[\big(\tilde{v}^{ij,*} - L^{ij}((v^{kl,*})_{k,l})\big)(t,x)\,;\quad \max\{\big(\tilde{v}^{ij,*} - U^{ij}((v^{kl,*})_{k,l})\big)(t,x),\\ \qquad -p - b(t,x)^\top q - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)M) - f^{ij}(t,x,[(v^{kl,*}(t,x))_{\substack{(k,l)\in\Gamma^1\times\Gamma^2\\(k,l)\neq(i,j)}}, \tilde{v}^{ij,*}(t,x)], \sigma^\top(t,x)q)\}] \leq 0 \end{cases}$$

where $[(v^{kl,*}(t,x))_{\substack{(k,l)\in\Gamma^1\times\Gamma^2\\(k,l)\neq(i,j)}}, \tilde v^{ij,*}(t,x)]$ is obtained from $(\tilde v^{kl,*}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2})$ by replacing $\tilde v^{ij,*}(t,x)$ with $v^{ij,*}(t,x)$. This means that $(t,x)\in[0,T)\times\mathbb{R}^k \longrightarrow \tilde v^{i,j}(t,x)$ is a subsolution of the following equation.

$$\begin{cases} \min[\big(w-L^{ij}((v^{kl,*})_{k,l})\big)(t,x), \quad \max\{\big(w-U^{ij}((v^{kl,*})_{k,l})\big)(t,x), \\ \qquad -p-b(t,x)^\top q-\dfrac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)M)-f^{ij}(t,x,[(v^{kl,*}(t,x))_{\substack{(k,l)\in\Gamma^1\times\Gamma^2\\(k,l)\neq(i,j)}}, w], \sigma^\top(t,x)q)\}] = 0. \end{cases}$$

Next, relying on the lower semi continuity of the function

$$\begin{cases} (t,x,w,p,q,M)\in[0,T]\times\mathbb{R}^{k+1+1+k}\times\mathbb{S}^k \longmapsto \quad \min\{\Big(w-\max_{k\neq i}(v^{kj,*}(t,x)-\underline{g}_{ik}(t,x))\Big), \\ \qquad \max[\big(w-\min_{l\neq j}(v^{il,*}(t,x)+\bar g_{jl}(t,x))\big), -p-b(t,x)^\top q-\dfrac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)M) \\ \qquad\qquad -f^{ij}(t,x,[(v^{kl,*}(t,x))_{\substack{(k,l)\in\Gamma^1\times\Gamma^2\\(k,l)\neq(i,j)}}, w], \sigma^\top(t,x)q)]\} \end{cases}$$

and using Lemma 4.2, in Crandall *et al.* (1992) ([5], pp.23), related to suprema of subsolutions, combined with the above result, it holds that v^{ij} is a subsolution of the following equation:

$$\begin{cases} \min[\big(w-L^{ij}((v^{kl,*})_{k,l})\big)(t,x), \quad \max\{\big(w-U^{ij}((v^{kl,*})_{k,l})\big)(t,x), \\ \qquad -p-b(t,x)^\top q-\dfrac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)M)-f^{ij}(t,x,[(v^{kl,*}(t,x))_{\substack{(k,l)\in\Gamma^1\times\Gamma^2\\(k,l)\neq(i,j)}}, w(t,x)], \sigma^\top(t,x)q)\}] = 0, \\ v^{ij}(T,x)=h^{ij}(x). \end{cases} \tag{5.18}$$

Since (i,j) is arbitrary in $\Gamma^1\times\Gamma^2$, $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a subsolution of (2.1).

Step 2. In this step we use the so called Perron's method to show that $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a viscosity supersolution of (2.1).

Indeed, we first note for any $(i,j)\in\Gamma^1\times\Gamma^2$,

$$\underline{v}^{ij}=\underline{v}^{ij}_*\le \bar v^{ij}_*\le v^{ij}_*\le \bar v^{ij,m_0}_*=\bar v^{ij,m_0},$$

since $\bar v^{ij,m_0}$ is continuous and $\underline{v}^{ij}$ is *lsc*. Therefore, for any $x\in\mathbb{R}^k$,

$$v^{ij}_*(T,x)=h^{ij}(x) \tag{5.19}$$

thanks to $\underline{v}^{ij}(T,x)=h^{ij}(x)=\bar v^{ij,m_0}(T,x)$. Next, assume that $(v^{i,j})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is not a supersolution for (2.1). Then, taking into account of (5.19) and Remark 1, there exists at least one pair (i,j) such that $v^{i,j}$ does not satisfy the viscosity supersolution property: this means that for some point $(t_0,\ x_0)\in[0,T)\times\mathbb{R}^k$ there exists a triple (p,q,M) in $\mathcal{J}^-(v^{ij}_*)(t_0,x_0)$ such that

$$\begin{cases} \min[\big(v^{ij}_*-L^{ij}((v^{kl}_*)_{(k,l)\in\Gamma^1\times\Gamma^2})\big)(t_0,x_0)\,; \max\{\big(v^{ij}_*-U^{ij}((v^{kl}_*)_{(k,l)\in\Gamma^1\times\Gamma^2})\big)(t_0,x_0), \\ \qquad -p-b(t_0,x_0)^\top q-\frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t_0,x_0)M)-f^{ij}(t_0,x_0,(v^{kl}_*(t_0,x_0))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t_0,x_0).q)\}]<0. \end{cases} \tag{5.20}$$

We now follow the same idea as in Crandall *et al.* (1992) ([5], pp.24). For any positive δ, γ and r, set $u_{\delta,\gamma}$ and B_r as follows:

$$u_{\delta,\gamma}(t,x)=v^{ij}_*(t_0,x_0)+\delta+\langle q,x-x_0\rangle+p(t-t_0)+\frac{1}{2}\langle(M-2\gamma)(x-x_0),(x-x_0)\rangle,$$

and

$$B_r:=\{(t,x)\in[0,T]\times\mathbb{R}^k,\ \text{s.t. } |t-t_0|+|x-x_0|<r\}.$$

Using (5.20) and continuity of all the data, choosing δ, γ small enough we obtain

$$\begin{cases} \min[\big(v^{ij}_*+\delta-L^{ij}((v^{kl}_*)_{(k,l)\in\Gamma^1\times\Gamma^2})\big)(t_0,x_0), \ \max\{\big(v^{ij}_*+\delta-U^{ij}((v^{kl}_*)_{(k,l)\in\Gamma^1\times\Gamma^2})\big)(t_0,x_0), \\ \qquad -p-b(t,x)^\top q-\frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t_0,x_0)(M-2\gamma)) \\ \qquad\quad -f^{ij}(t_0,x_0,[(v^{kl}_*(t_0,x_0))_{(k,l)\neq(i,j)}, v^{ij}_*(t_0,x_0)+\delta],\sigma^\top(t_0,x_0)q)\}]<0. \end{cases} \tag{5.21}$$

Next, let us define the function Υ as follows.

$$\Upsilon(t,x)= \min\left\{u_{\delta,\gamma}(t,x)-\max_{k\neq i}\big(v_*^{kj}-\underline{g}_{ik}\big)(t,x),\ \max\{u_{\delta,\gamma}(t,x)-\min_{l\neq j}\big(v_*^{il}+\bar{g}_{jl}\big)(t,x),\ \varpi(t,x)\}\right\},$$

where,

$$\varpi(t,x)=-p-b(t,x)^\top q-\frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)(M-2\gamma))-f^{ij}(t,x,[(v_*^{kl}(t,x))_{(k,l)\neq(i,j)},u_{\delta,\gamma}(t,x)],\sigma^\top(t,x)q).$$

First we note that from (5.21), $\Upsilon(t_0,x_0)<0$, since $u_{\delta,\gamma}(t_0,x_0)=v_*^{ij}(t_0,x_0)+\delta$. On the other hand by the continuity of $u_{\delta,\gamma}$, Assumptions (H1)-(H2) on f^{ij} and finally the lower semi-continuity of v_*^{kl}, $(k,l)\in\Gamma^1\times\Gamma^2$, we can easily check that the function Υ is *usc*. Thus, for any $\varepsilon>0$, there exists $\eta>0$ such that for any $(t,x)\in B_\eta$ we have $\Upsilon(t_0,x_0)\geq\Upsilon(t,x)-\epsilon$. Since $\Upsilon(t_0,x_0)<0$, then choosing ε small enough we deduce that $\Upsilon(t,x)\leq 0$ for any $(t,x)\in B_\eta$ with an appropriate η. It follows that the function $u_{\delta,\gamma}$ is a viscosity subsolution on B_η of the following system.

$$\begin{aligned}\min\Big\{&\varrho(t,x)-\max\nolimits_{k\neq i}\big(v_*^{kj}-\underline{g}_{ik}\big)(t,x),\ \max\{\varrho(t,x)-\min\nolimits_{l\neq j}\big(v_*^{il}+\bar{g}_{jl}\big)(t,x),\\ &-\partial_t\varrho(t,x)-b(t,x)^\top D_x\varrho(t,x)-\tfrac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)D^2_{xx}\varrho(t,x))\\ &\quad -f^{ij}(t,x,[(v_*^{kl}(t,x))_{(k,l)\neq(i,j)},\varrho(t,x)],\sigma^\top(t,x)D_x\varrho(t,x))\}\Big\}=0.\end{aligned}$$

Since, for any $(k,l)\in\Gamma^1\times\Gamma^2$, $v_*^{kl}\leq v^{kl,*}$ and since f^{ij} satisfies the monotonicity condition (H2), $u_{\delta,\gamma}$ is also a viscosity subsolution on B_η of the system

$$\begin{aligned}\min\Big\{&\varrho(t,x)-\max\nolimits_{k\neq i}\big(v^{kj,*}-\underline{g}_{ik}\big)(t,x);\ \max\{\varrho(t,x)-\min\nolimits_{l\neq j}\big(v^{il,*}+\bar{g}_{jl}\big)(t,x),\\ &-\partial_t\varrho(t,x)-b(t,x)^\top D_x\varrho(t,x)-\tfrac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)D^2_{xx}\varrho(t,x))\\ &\quad -f^{ij}(t,x,[(v^{kl,*}(t,x))_{(k,l)\neq(i,j)},\varrho(t,x)],\sigma^\top(t,x)D_x\varrho(t,x))\}\Big\}=0\end{aligned}\tag{5.22}$$

Next, as $(p,q,M)\in\mathcal{J}^-(v_*^{ij}(t_0,x_0))$ then

$$\begin{aligned}v^{ij}(t,x)\geq v_*^{ij}(t,x)&\geq v_*^{ij}(t_0,x_0)+p(t-t_0)+\langle q,x-x_0\rangle\\ &+\tfrac{1}{2}\langle M(x-x_0),(x-x_0)\rangle+o(|t-t_0|)+o(|x-x_0|^2).\end{aligned}$$

In view of the definition of $u_{\delta,\gamma}$ and taking $\delta=\frac{r^2}{8}\gamma$, it is easily seen that

$$v^{ij}(t,x)>u_{\delta,\gamma}(t,x),$$

as soon as $\frac{r}{2}<|x-x_0|\leq r$ and r small enough. We now take $r\leq\eta$ and consider the function $\tilde{u}^{ij}$:

$$\tilde{u}^{ij}(t,x)=\begin{cases}\max(v^{ij}(t,x),\ u_{\delta,\gamma}(t,x)), & \text{if } (t,x)\in B_r,\\ v^{ij}(t,x), & \text{otherwise.}\end{cases}$$

Then taking into account of (5.22) and using Lemma 4.2 in Crandall *et al.* (1992) ([5]), it follows that $\tilde{u}^{ij}$ is a subsolution of (5.18). Next, as $\tilde{u}^{ij}\geq v^{ij}$ and using once more the monotonicity assumption (H2) on f^{kl}, $(k,l)\in\Gamma^1\times\Gamma^2$, we get that $[(v^{kl})_{(k,l)\neq(i,j)},\tilde{u}^{ij}]$ is also a subsolution of (2.1) which belongs to Π_g. Thus, thanks to the Comparison Theorem 4.2, $[(v^{kl})_{(k,l)\neq(i,j)},\tilde{u}^{ij}]$ belongs also to $\mathcal{U}_{m_0}$. Finally, in view of the definition of v_*^{ij}, there exists a sequence $(t_n,x_n,v^{ij}(t_n,x_n))_{n\geq1}$ that converges to $(t_0,x_0,v_*^{ij}(t_0,x_0))$. This implies that

$$\lim_n(\tilde{u}^{ij}-v^{ij})(t_n,x_n)=\big(u_{\delta,\gamma}-v_*^{ij}\big)(t_0,x_0)=v_*^{ij}(t_0,x_0)+\delta-v_*^{ij}(t_0,x_0)>0.$$

This means that there are points (t_n,x_n) such that $\tilde{u}^{ij}(t_n,x_n)>v^{ij}(t_n,x_n)$. But this contradicts the definition of v^{ij}, since $[(v^{kl})_{(k,l)\neq(i,j)},\tilde{u}^{ij}]$ belongs also to $\mathcal{U}_{m_0}$. Therefore, $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is also a supersolution for (2.1).

Now, in view of Corollary 2, $({}^{m_0}v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the unique continuous viscosity solution in the class Π_g of (2.1). Thus, using once more uniqueness, we deduce that $({}^{m_0}v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ does not depend on m_0. ■

As above, let us denote by $(v^{i,j})_{(i,j)\in\Gamma^1\times\Gamma^2}$ the family $({}^{m_0}v^{i,j})_{(i,j)\in\Gamma^1\times\Gamma^2}$. Here is the second main result of the paper.

Theorem 5.4. *For any $(i,j)\in\Gamma^1\times\Gamma^2$, $\bar{v}^{ij}=v^{ij}$, i.e., $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is continuous and is the unique viscosity solution in the class Π_g of the system (2.1).*

Proof. For any $(i,j)\in\Gamma^1\times\Gamma^2$ and $m_0\in\mathbb{N}$ we have

$$\bar{v}^{ij}\le v^{ij}\le \bar{v}^{ij,m_0}.$$

Taking the limit as $m_0\to\infty$ we obtain $\bar{v}^{ij}=v^{ij}$, for all $(i,j)\in\Gamma^1\times\Gamma^2$. Finally, Theorem 5.3 completes the proof. ■

As a by-product of this result, we have the following theorem for the family $(\underline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$.

Theorem 5.5. *The functions $(\underline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ are continuous, of polynomial growth and unique viscosity solution in the class Π_g of the following system of variational inequalities. For every $(i,j)\in\Gamma^1\times\Gamma^2$,*

$$\begin{cases}\max\Big\{\underline{v}^{ij}(t,x)-\min_{l\in(\Gamma^2)^{-j}}(\underline{v}^{il}+\bar{g}_{jl})(t,x),\ \min\Big\{\underline{v}^{ij}(t,x)-\max_{k\in(\Gamma^1)^{-i}}(\underline{v}^{kj}-\underline{g}_{ik})(t,x),\\ \qquad -\partial_t\underline{v}^{ij}(t,x)-\mathcal{L}\underline{v}^{ij}(t,x)-f^{ij}(t,x,(\underline{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x)D_x\underline{v}^{ij}(t,x))\Big\}\Big\}=0\\ \underline{v}^{ij}(T,x)=h^{ij}(x).\end{cases}\tag{5.23}$$

Proof. It is enough to consider $(-\underline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ which, in view of Theorem 5.4, is continuous, of polynomial growth and the unique viscosity solution of the following system. For all $(i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases}\min\Big\{v^{ij}(t,x)-\max_{l\in(\Gamma^2)^{-j}}(v^{il}-\bar{g}_{jl})(t,x),\ \max\Big\{v^{ij}(t,x)-\min_{k\in(\Gamma^1)^{-i}}(v^{kj}+\underline{g}_{ik})(t,x),\\ \qquad -\partial_t v^{ij}(t,x)-\mathcal{L}v^{ij}(t,x)+f^{ij}(t,x,(-v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},-\sigma^\top(t,x)D_x v^{ij}(t,x))\Big\}\Big\}=0\\ v^{ij}(T,x)=-h^{ij}(x).\end{cases}\tag{5.24}$$

Using now the result by Barles ([2], pp. 18), we obtain that $(\underline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ are continuous, of polynomial growth and unique viscosity solution in the class Π_g of system (5.23), which is the desired result. ■

Remark 2. *We do not know whether or not we have $\underline{v}^{ij}=\bar{v}^{ij}$, $(i,j)\in\Gamma^1\times\Gamma^2$.*

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6 Appendix

A1. *Comparison of solutions of multi-dimensional BSDEs* ([22], Theorem 1, pp.135)

Theorem 6.1. *Let (Y,Z) (resp. $(\bar{Y},\bar{Z})$) be the solution of the k-dimensional BSDE associated with $(f := (f_i(t,\omega,y,z))_{i=1,k},\xi)$ (resp. $(\bar{f} := (\bar{f}_i(t,\omega,y,z))_{i=1,k},\bar{\xi})$) where:*

(i) ξ and $\bar{\xi}$ are square integrable $\mathcal{F}_T$-random variables of $\mathbb{R}^k$;

(ii) the functions $f(t,\omega,y,z)$ and $\bar{f}(t,\omega,y,z)$ defined on $[0,T]\times\Omega\times\mathbb{R}^{k+k\times d}$ are $\mathbb{R}^k$-valued, Lipschitz in (y,z) uniformly in (t,ω) and the process $(f(t,\omega,0,0))_{t\le T}$ (resp. $(\bar{f}(t,\omega,0,0))_{t\le T}$) belongs to $\mathcal{H}^{2,k}$;

(iii) for any $i=1,\dots,k$, the $i-th$ component f_i (resp. $\bar{f}_i$) of f (resp. $\bar{f}$) depends only on the $i-th$ row of the matrix z.

If there exists a constant $C\ge 0$, such that for any $y,\bar{y}\in\mathbb{R}^k$, $z,\bar{z}\in\mathbb{R}^{k\times d}$

$$-4\sum_{i=1,k} y_i^-(f_i(t,\omega,y_1^+ +\bar{y}_1,\dots,y_k^+ +\bar{y}_k,z)-\bar{f}_i(t,\omega,\bar{y}_1,\dots,\bar{y}_k,\bar{z}))\le 2\sum_{i=1,k}\mathbb{1}_{[y_i<0]}|z-\bar{z}|^2+C\sum_{i=1,k}(y_i^-)^2$$

where $y_i^+=max(y_i,0)$ and $y_i^-=max(-y_i,0)$. Then for any $i=1,\dots,k$, $\mathbb{P}$-a.s., $Y^i\le\bar{Y}^i$. ■

A2. *Systems of reflected BSDEs with one inter-connected barrier and their related systems of variational inequalities* (see e.g. [17] or [19]).

Let $\mathcal{J} := \{1,\dots,m\}$ and let us consider the following functions: for $i,j\in\mathcal{J}$,

$$\begin{array}{l} f_i:\ (t,x,y^1,\dots,y^m,z)\in[0,T]\times\mathbb{R}^{k+m+d}\mapsto f_i(t,x,y^1,\dots,y^m,z)\ \in\mathbb{R}\,;\\ g_{ij}:\ (t,x)\in[0,T]\times\mathbb{R}^k\mapsto g_{ij}(t,x)\in\mathbb{R}\,(i\neq j);\\ h_i: x\in\mathbb{R}^k\mapsto h_i(x)\in\mathbb{R}. \end{array}$$

We now make the following assumptions.

[Af]. For $i\in\mathcal{J}$, f_i satisfies

(i) The function $(t,x)\mapsto f_i(t,x,y^1,\dots,y^m,z)$ is continuous uniformly w.r.t. $(\overrightarrow{y},z):=(y^1,\dots,y^m,z)$.

(ii) The function f_i is uniformly Lipschitz continuous with respect to $(\overrightarrow{y},z):=(y^1,\dots,y^m,z)$, i.e., for some $C\geq 0$,

$$|f_i(t,x,y^1,\dots,y^m,z)-f_i(t,x,\bar{y}^1,\dots,\bar{y}^m,\bar{z})|\leq C(|y^1-\bar{y}^1|+\cdots+|y^m-\bar{y}^m|+|z-\bar{z}|).$$

(iii) The mapping $(t,x)\mapsto f_i(t,x,0,\dots,0)$ is $\mathcal{B}([0,T]\times\mathbb{R}^k)$-measurable and of polynomial growth i.e. it belongs to Π_g;

(iv) Monotonicity. $\forall i\in\mathcal{J}$, for any $k\in\mathcal{J}^{-i}$, the mapping $y_k\in\mathbb{R}\mapsto f_i(t,x,y_1,\dots,y_{k-1},y_k,y_{k+1},\dots,y_m)$ is non-decreasing whenever the other components $(t,x,y_1,\dots,y_{k-1},y_{k+1},\dots,y_m)$ are fixed.

[Ag]. (i) The function g_{ij} is jointly continuous in (t,x), non-negative, i.e., $g_{ij}(t,x)\geq 0$, $\forall (t,x)\in[0,T]\times\mathbb{R}^k$ and belongs to Π_g.

(ii) *The no free loop property.* for any $(t,x)\in[0,T]\times\mathbb{R}^k$ and for any sequence of indexes $i_1,\dots,i_k$ such that $i_1=i_k$ and $card\{i_1,\dots,i_k\}=k-1$ we have

$$g_{i_1i_2}(t,x)+g_{i_2i_3}(t,x)+\cdots+g_{i_{k-1}i_k}(t,x)+g_{i_ki_1}(t,x)>0,\ \forall (t,x)\in[0,T]\times\mathbb{R}^k.$$

As a convention we assume hereafter that $g_{ii}(t,x)=0$ for any $(t,x)\in[0,T]\times\mathbb{R}^k$ and $(i,j)\in\Gamma^1\times\Gamma^2$.

[Ah1]. h_i is continuous, belongs to Π_g and satisfies:

$$\forall x\in\mathbb{R},\ h_i(x)\geq\max_{j\in\mathcal{J}^{-i}}(h_j(x)-g_{ij}((T,x)).$$

[Ah2]. The function h_i is continuous, belongs to Π_g and satisfies

$$\forall x\in\mathbb{R},\ h_i(x)\geq\min_{j\in\mathcal{J}^{-i}}(h_j(x)+g_{ij}((T,x)).$$

Then, we have

Theorem 6.2. *Assume that [Ah], [Ag] and [Ah1] are fulfilled. Then, there exist m triples of processes $((Y^{i;t,x},Z^{i;t,x},K^{i;t,x}))_{i\in\mathcal{J}}$ that satisfy: $\forall\ i\in\mathcal{J}$,*

$$\begin{cases} Y^i,K^i\in\mathcal{S}^2,\ Z^i\in\mathcal{H}^{2,d},\ K^i \text{ non-decreasing and } K^i_0=0;\\ Y^i_s=h_i(X^{t,x}_T)+\int_s^T f_i(r,X^{t,x}_r,Y^1_r,\dots,Y^m_r,Z^i_r)dr+K^i_T-K^i_s-\int_s^T Z^i_rdB_r,\ \forall\ s\leq T\\ Y^i_s\geq\max_{j\in\mathcal{J}^{-i}}\{Y^j_s-g_{ij}(s,X^{t,x}_s)\},\ \forall s\leq T\\ \int_0^T(Y^i_s-\max_{j\in\mathcal{J}^{-i}}\{Y^j_s-g_{ij}(s,X^{t,x}_s)\})dK^i_s=0.\end{cases}\tag{6.1}$$

Moreover there exist m deterministic functions $(v^i(t,x))_{i\in\mathcal{J}}$ continuous and belonging to Π_g such that:

$$\forall s\in[t,T],Y^{i;t,x}_s=v^i(s,X^{t,x}_s).$$

Finally $(v^i(t,x))_{i\in\mathcal{J}}$ is the unique solution, in the sub-class of Π_g of continuous functions, of the following system of variational inequalities with inter-connected obstacles: $\forall\ i\in\mathcal{J}$

$$\begin{cases}\min\Big\{v_i(t,x)-\max\limits_{j\in\mathcal{J}^{-i}}(-g_{ij}(t,x)+v_j(t,x)),\\ \qquad -\partial_tv_i(t,x)-\mathcal{L}v_i(t,x)-f_i(t,x,v^1(t,x),\dots,v^m(t,x),\sigma^\top(t,x)D_xv^i(t,x))\Big\}=0\ ;\\ v_i(T,x)=h_i(x).\end{cases}\tag{6.2}$$

Remark 3. *In equations (6.1) and (6.2), if instead we have required an upper barrier reflection, then one would have obtained a similar result which can be stated as follows.*

Assume that [Af], [Ag] and [Ah2] are fulfilled. Then there exist m triples of processes $((\tilde{Y}^{i;t,x}, \tilde{Z}^{i;t,x}, \tilde{K}^{i;t,x}))_{i\in\mathcal{J}}$ that satisfy, for all $i \in \mathcal{J}$,

$$\begin{cases} \tilde{Y}^i, \tilde{K}^i \in \mathcal{S}^2,\ \tilde{Z}^i \in \mathcal{H}^{2,d},\ \tilde{K}^i \text{ non-decreasing and } \tilde{K}^i_0 = 0; \\ \tilde{Y}^i_s = h_i(X^{t,x}_T) + \int_s^T f_i(r, X^{t,x}_r, \tilde{Y}^1_r, \ldots, \tilde{Y}^m_r, \tilde{Z}^i_r)dr - (\tilde{K}^i_T - \tilde{K}^i_s) - \int_s^T \tilde{Z}^i_r dB_r, \quad t \le s \le T, \\ \tilde{Y}^i_s \le \min_{j\in\mathcal{J}^{-i}}\{\tilde{Y}^j_s + g_{ij}(s, X^{t,x}_s)\}, \quad t \le s \le T, \\ \int_0^T (\tilde{Y}^i_s - \min_{j\in\mathcal{J}^{-i}}\{\tilde{Y}^j_s + g_{ij}(s, X^{t,x}_s)\})d\tilde{K}^i_s = 0. \end{cases} \tag{6.3}$$

Moreover, there exist m deterministic functions $(\tilde{v}^i(t,x))_{i\in\mathcal{J}}$ continuous and belong to Π_g such that:

$$\tilde{Y}^{i;t,x}_s = \tilde{v}^i(s, X^{t,x}_s), \qquad s \in [t,T].$$

Finally, $(\tilde{v}^i(t,x))_{i\in\mathcal{J}}$ is the unique solution, in the subclass of Π_g of continuous functions, of the following system of variational inequalities with interconnected obstacles. For all $i \in \mathcal{J}$

$$\begin{cases} \max\Big\{\tilde{v}_i(t,x) - \min_{j\in\mathcal{J}^{-i}}(g_{ij}(t,x) + \tilde{v}_j(t,x)), \\ \qquad -\partial_t \tilde{v}_i(t,x) - \mathcal{L}\tilde{v}_i(t,x) - f_i(t,x,\tilde{v}^1(t,x), \ldots, \tilde{v}^m(t,x), \sigma^\top(t,x)D_x\tilde{v}^i(t,x))\Big\} = 0\ ; \\ \tilde{v}_i(T,x) = h_i(x). \end{cases} \tag{6.4}$$

The proof of this result is obtained straightforward from Theorem 6.2, in considering the equations satisfied by $((-\tilde{Y}^i, -\tilde{Z}^i, \tilde{K}^i))_{i\in\mathcal{J}}$.

A3. *Linearization procedure of Lipschitz functions.* Let f be a function from $\mathbb{R}^2$ to $\mathbb{R}$ which with (x_1, x_2) associates $f(x_1, x_2)$ which is Lipschitz in its arguments. Then, we can write

$$\begin{aligned} f(x_1,x_2) - f(y_1,y_2) &= f(x_1,x_2) - f(y_1,x_2) + f(y_1,x_2) - f(y_1,y_2) \\ &= 1\!\!1_{x_1-y_1\neq 0}\tfrac{f(x_1,x_2)-f(y_1,x_2)}{x_1-y_1}(x_1-y_1) + 1\!\!1_{x_2-y_2\neq 0}\tfrac{f(y_1,x_2)-f(y_1,y_2)}{x_2-y_2}(x_2-y_2) \\ &= a_1(x_1,x_2,y_1).(x_1-y_1) + a_2(x_2,y_1,y_2).(x_2-y_2) \end{aligned} \tag{6.5}$$

where, a_1 and a_2 are measurable functions and bounded, i.e.,

$$|a_1(x_1,x_2,y_1)| \vee |a_2(x_2,y_1,y_2)| \le C(f),\ i = 1,2,$$

where, $C(f)$ is the Lipschitz constant of f. Moreover, if f is non-decreasing with respect to x_1 (resp. x_1) when x_2 (resp. x_1) is fixed, then $a_i \ge 0$, $i = 1,2$.

Linearizing f consists of writing f as

$$f(x_1,x_2) - f(y_1,y_2) = a_1(x_1,x_2,y_1).(x_1-y_1) + a_2(x_2,y_1,y_2).(x_2-y_2).$$

A4. *Representation of a penalization scheme of two barriers reflected BSDE.*

For $n \ge 0$ let $(Y^n, Z^n, K^{+,n})$ be the solution of the following one barrier reflected BSDE.

$$\begin{cases} (Y^n, Z^n, K^{+,n}) \in \mathcal{S}^{2,1} \times \mathcal{H}^{2,d} \times \mathcal{A}^{+,2} \\ Y^n_t = \xi + \int_t^T g(s)ds - n\int_t^T (Y^n_s - U_s)^+ ds + K^{+,n}_T - K^{n,+}_t - \int_t^T Z^n_s dB_s, \quad t \le T; \\ Y^n_t \ge L_t, \quad 0 \le t \le T \ \text{ and } \ \int_0^T (Y^n_s - L_s)dK^{+,n}_s = 0 \end{cases} \tag{6.6}$$

where, the processes L and U belong to $\mathcal{S}^{2,1}$, $(g(s))_{s\leq T} \in \mathcal{H}^{2,1}$, ξ is square integrable and $\mathcal{F}_T$-measurable. Moreover, we require that $L \leq U$ and $L_T \leq \xi$. Under these conditions, the solution $(Y^n, Z^n, K^{+,n})$ exists and is unique (see e.g. [12]). Next, for $n \geq 0$ and $t \leq T$, set $K_t^{n,-} = n\int_0^t (Y_s^n - U_s)^+ ds$. Then, $K^{-,n} \in \mathcal{A}^{+,2}$ and $\int_0^T (Y_s^n - Y_s^n \vee U_s) dK_s^{-,n} = 0$. Therefore, the equation (6.6) can be expressed as a BSDE with two reflecting barriers in the following manner. For all $t \leq T$,

$$\begin{cases} Y_t^n = \xi + \int_t^T g(s)ds + (K_T^{+,n} - K_t^{+,n}) - (K_T^{-,n} - K_t^{-,n}) - \int_t^T Z_s^n dB_s; \\ L_t \leq Y_t^n \leq Y_t^n \vee U_t, \\ \int_0^T (Y_s^n - L_s) dK_s^{+,n} = \int_0^T (Y_s^n - Y_s^n \vee U_s) dK_s^{-,n} = 0. \end{cases} \tag{6.7}$$

Thus, a result by Cvitanic and Karatzas [6] or Hamadène and Lepeltier [16] allows to represent Y^n as a value function of a Dynkin game, i.e., it holds true that for any $t \leq T$,

$$\begin{aligned} Y_t^n &= \text{ess sup}_{\sigma\geq t}\text{ess inf}_{\tau\geq t}\mathbb{E}[\int_t^{\sigma\wedge\tau} g(s)ds + L_\sigma \mathbb{1}_{[\sigma<\tau]} + (Y_\tau^n \vee U_\tau)\mathbb{1}_{[\tau\leq\sigma<T]} + \xi\mathbb{1}_{[\tau=\sigma=T]}|\mathcal{F}_t] \\ &= \text{ess inf}_{\tau\geq t}\text{ess sup}_{\sigma\geq t}\mathbb{E}[\int_t^{\sigma\wedge\tau} g(s)ds + L_\sigma \mathbb{1}_{[\sigma<\tau]} + (Y_\tau^n \vee U_\tau)\mathbb{1}_{[\tau\leq\sigma<T]} + \xi\mathbb{1}_{[\tau=\sigma=T]}|\mathcal{F}_t]. \end{aligned}$$

where τ and σ are $\mathcal{F}$-stopping times.